NOTE ON THE DISCRIMINANT MATRIX OF AN ALGEBRA*

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The purpose of this note is to extend MacDuffee's normal basis† to a general linear associative algebra.

Let $\mathfrak{A}$ be a linear associative algebra over an infinite field $\mathfrak{F}$, with the basis $e_1, e_2, \cdots, e_n$, and let the constants of multiplication be denoted by c_{ijk}. Let $T_1 = (\tau_{rs})$ be the first discriminant matrix of $\mathfrak{A}$, and let $d_h = \sum_k c_{hkk}$. Then $\tau_{rs} = \tau_{sr} = \sum_h c_{srh} d_h$.

If $\mathfrak{A}$ is nilpotent, $d_i = 0$, $(i = 1, 2, \cdots, n)$,‡ and $T_1 = 0$. We now suppose that $\mathfrak{A}$ is non-nilpotent and therefore possesses a principal idempotent element e_1.§ Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$, and $\mathfrak{B}$ be the set of elements x of $\mathfrak{A}$ for which $e_1 x = 0$. Then $\mathfrak{B} < \mathfrak{N}$.¶ It is easily shown that $\mathfrak{A} = e_1\mathfrak{A} + \mathfrak{B}$, where $e_1\mathfrak{A}$ and $\mathfrak{B}$ are algebras whose intersection is zero. Let $e_1\mathfrak{A} = \mathfrak{L} + \overline{\mathfrak{N}}$, where $\overline{\mathfrak{N}}$ is the radical of $e_1\mathfrak{A}$ and $\mathfrak{L}$ is a linear system supplementary to $\overline{\mathfrak{N}}$ in $e_1\mathfrak{A}$. It is not difficult to show that $\mathfrak{N} = \overline{\mathfrak{N}} + \mathfrak{B}$.‖ We may therefore select the basis of $\mathfrak{A}$ as $e_1, e_2, \cdots, e_n$, so that e_1 is the principal idempotent selected above, $e_1, e_2, \cdots, e_\sigma$ is a basis for $\mathfrak{L}$, $e_{\sigma+1}, e_{\sigma+2}, \cdots, e_\rho$ a basis for $\overline{\mathfrak{N}}$, and $e_{\rho+1}, e_{\rho+2}, \cdots, e_n$ a basis for $\mathfrak{B}$. Then $d_i = 0, (i > \sigma)$,** and $d_1 = \sum_k c_{1kk} = \rho > 0$, since if x is in $e_1\mathfrak{A}$, we have $e_1 x = x$.

Direct computation shows that if $e_1, e_2, \cdots, e_n$ are subjected to a transformation, $e_i' = \sum_j a_{ij} e_j$, the new d's are given by $d_i' = \sum_j a_{ij} d_j$, $(i = 1, 2, \cdots, n)$. Hence if we make the non-singular transformation

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† C. C. MacDuffee, Transactions of this Society, vol. 33, p. 427, proves Theorems 1 and 2 only for algebras with a principal unit. The terminology and notation in this paper are in agreement with that of MacDuffee.

‡ L. E. Dickson, *Algebren und ihre Zahlentheorie*, 1927, p. 108.

§ Dickson, loc. cit., p. 100.

¶ Dickson, loc. cit., p. 100.

‖ This relation follows directly from Dickson, loc. cit., p. 100, Theorem 5, or it can be proved independently.

** Dickson, loc. cit., p. 108.

$$\begin{cases} e_1' = e_1, & \\ e_i' = -\dfrac{d_i}{\rho} e_1 + e_i, & (1 < i \leqq \sigma), \\ e_i' = e_i, & (i > \sigma), \end{cases}$$

we obtain $d_1' = \rho$, $d_i' = 0$, $(i > 1)$. This transformation does not alter the bases of $\overline{\mathfrak{N}}$ and $\mathfrak{B}$.

We now have $\tau_{11}' = d_1' = \rho$, $\tau_{r1}' = \tau_{1r}' = c_{1r1}' d_1' = 0$, $(r > 1)$ and, since $\mathfrak{N}$ is an invariant subalgebra of $\mathfrak{A}$, $c_{ijk}' = 0$, $(i \text{ or } j > \sigma, k \leqq \sigma)$, and therefore $\tau_{rs}' = \tau_{sr}' = c_{sr1}' d_1 = 0$, $(r$ or $s > \sigma)$. This gives

$$T_1' = \begin{Vmatrix} \rho & 0 & 0 & \cdots 0 & 0 \cdots 0 \\ 0 & \tau_{22}' & \tau_{23}' & \cdots \tau_{2\sigma}' & 0 \cdots 0 \\ 0 & \tau_{32}' & \tau_{33}' & \cdots \tau_{3\sigma}' & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdots & \cdots \\ 0 & \tau_{\sigma 2}' & \tau_{\sigma 3}' & \cdots \tau_{\sigma\sigma}' & 0 \cdots 0 \\ 0 & 0 & 0 & \cdots 0 & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdots & \cdots \\ 0 & 0 & 0 & \cdots 0 & 0 \cdots 0 \end{Vmatrix}.$$

It is obvious that T_1' can now be reduced to a diagonal matrix by transformations in $\mathfrak{F}$ which leave e_1', $e_{\sigma+1}'$, $e_{\sigma+2}'$, $\cdots$, e_n' invariant, and leave $d_2' = d_3' = \cdots = d_n' = 0$.

We may now reduce the basis of $\mathfrak{N}$ (or if $\mathfrak{A}$ is nilpotent, the basis of $\mathfrak{A}$ itself) to normal form* by a transformation in $\mathfrak{F}$ of the type

$$\begin{cases} e_i' = e_i, & (i \leqq \sigma), \\ e_i' = \displaystyle\sum_{j=\sigma+1}^{n} a_{ij} e_j, & (i > \sigma). \end{cases}$$

Such a transformation does not alter d_i, $(i = 1, 2, \cdots, n)$, e_1, or T_1.

Since the rank of T_1 is σ,† we have proved the following result.

THEOREM. *A basis can be so chosen for* $\mathfrak{A}$ *that*

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* Dickson, loc. cit., p. 111.

† Dickson, loc. cit., p. 110.

$$T_1 = \left\| \begin{matrix} g_1 & 0 & 0 \cdots 0 & 0 \cdots 0 \\ 0 & g_2 & 0 \cdots 0 & 0 \cdots 0 \\ 0 & 0 & g_3 \cdots 0 & 0 \cdots 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 \cdots g_\sigma & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 & 0 \cdots 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 \cdots 0 & 0 \cdots 0 \end{matrix} \right\|,$$

where the g's are in $\mathfrak{F}$ *and* $d_2 = d_3 = \cdots = d_n = 0$. *If* $\mathfrak{A}$ *is nilpotent, the basis is normal. If* $\mathfrak{A}$ *is not nilpotent,* $d_1 = g_1 \neq 0$, $g_i \neq 0$, $(i = 2, 3, \cdots, \sigma)$, *where* $n - \sigma$ *is the order of the radical of* $\mathfrak{A}$, *and* $e_{\sigma+1}, e_{\sigma+2}, \cdots, e_n$ *is a normal basis for this radical, and* e_1 *is a principal idempotent of* $\mathfrak{A}$.

We may now define a basis of the type whose existence is shown in the above theorem as a *normal basis for* $\mathfrak{A}$. In case $\mathfrak{A}$ is nilpotent, this basis is the ordinary normal basis for a nilpotent algebra; in case $\mathfrak{A}$ has a principal unit, it is MacDuffee's normal basis.

It is evident that a transformation of the form

$$\begin{cases} e_1' = e_1, & \\ e_i' = e_i + \sum_{j=\sigma+1}^{n} a_{ij} e_j, & (1 < i \leqq \sigma), \\ e_i' = e_i, & (i > \sigma), \end{cases}$$

leaves unaltered all the properties of the normal basis. But by such a transformation we can make $e_1, e_2, \cdots, e_\sigma$ the basis of a semi-simple subalgebra of $\mathfrak{A}$ having the principal unit e_1.*

COROLLARY. *The normal basis for a non-nilpotent algebra* $\mathfrak{A}$ *can be so chosen that* $(e_1, e_2, \cdots, e_\sigma)$ *is a semi-simple subalgebra of* $\mathfrak{A}$ *having the principal unit* e_1, *and* $(e_{\sigma+1}, e_{\sigma+2}, \cdots, e_n)$ *is the radical of* $\mathfrak{A}$.

As a consequence of the above theorem we can now omit from MacDuffee's Theorem 2 the restriction "*with a principal unit*".

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* Dickson, loc. cit., p. 136.