TWO BOOKS ON FOUNDATIONS OF GEOMETRY

Grundlagen der Geometrie. By Gerhard Hessenberg. Edited by W. Schwan. Berlin, de Gruyter (Göschens Lehrbücherei, 1. Gruppe, Band 17), 1930. 143 pp., 77 figs.

Vorlesungen über Grundlagen der Geometrie. By Kurt Reidemeister. Berlin, Springer (Die Grundlehren der mathematischen Wissenschaften, vol. 32), 1930. x+147 pp., 37 figs.

It is well that the simultaneous publication of two books on the bases of geometry, both by most competent authors, both making noteworthy contributions to the subject, should remind us of the progress which still continues after so many centuries. In writing on this topic, an author chooses between the exclusive use of strict logic, on the one hand, and, on the other, enough of an appeal to the pictorial sense to illuminate the subject and make it more accessible to readers with but a moderate power of abstraction. Reidemeister's book is of the first type; while that which was begun by Hessenberg and, after his death, completed by Schwan, is rather of the other kind.

The latter book, to be sure, almost completely satisfies a logician's demands, but there are a few diverting gaps in the argument—probably the result of divided authorship. Thus (p. 33), when the nature of lines and their segments has been carefully discussed, parallelism and direction are introduced with no foundation, no mention that there is such a thing as a plane. On page 77 we find the first use of the word "motion,"—"Let a plane E, which coincides with A, be so moved . . . ," and there is no hint of the concept of motion or its relation to congruence.

Distinctly good is the discussion of the logical relations between the Pascal and Desargues theorems, axioms of continuity (particularly that of Archimedes), order, incidence and parallelism, and the fundamental theorems of plane and solid projective geometry. There is, of course, Hessenberg's proof that Desargues' Theorem is a consequence of Pascal's. Important "artificial" geometries illustrate certain independences. Thus we have Moulton's non-desarguean plane geometry and various non-archimedean geometries. The latter group of examples shows how the axioms of incidence, order and parallelism are insufficient to prove Pascal's Theorem, and how the axioms named, plus this theorem, are still not enough to establish the axiom of Archimedes. Since Pascal's Theorem, or its corollary, Desargues', is sufficient for the transition from two dimensions to more, the problem is proposed of discovering what axiom, weaker than the archimedean, will suffice to establish this transition without the use of congruence.

After a short, clear initial chapter on the concepts of equality, order, and continuity, that of congruence leads up to a discussion of numerical measurement. Hessenberg treats measurement of segments, Schwan continues with that of vectors—with the gap in definitions already noted. In the measurement of angles, Schwan distinguishes between the angles themselves (figures consisting of two half-lines and, in measure, never exceeding π) and angular fields (portions of the plane swept over in possible motion from one side to the other). Angles, like complex numbers of absolute value 1, are combined by "multiplication," fields by "addition." There is some convenience, from the point of view of elementary geometry, in having a pair of half-lines furnish but one angle; yet there is awkwardness also, and it would seem simpler to be content with "equality" and "congruence" of unrestricted angles. We can hardly agree that "the multiplication of two obtuse angles yields an acute one." Finally, Hessenberg treats adequately of the measurement of plane areas (with emphasis on Hilbert's work on non-archimedean geometry) and of solids.

Chapter 3 treats of projective geometry in the plane. Besides the emphasis on the logical relation between the basic theorems and axioms, we find the clever schematization of the Pascal configuration very pleasing. It is only at the close, by the introduction of ideal points, that the plane truly becomes projective; the requirement of continuity is avoided, as is frequently the case in this book, the device here being the use of reflections and motions in the definition of ideal points.

A noteworthy feature of the treatment of spatial projective geometry is that "artificial" geometry of Hessenberg which interprets each point in space as a triangle in a fixed plane whose sides have given directions. This device, suggested by descriptive geometry, elegantly illustrates the desarguean transition from plane to solid geometry. Coordinates on a line are set up in a method whose distinctive feature is the use of a particular line at infinity.

In both the books here reviewed there is interesting and enlightening comment on relations between geometries on divers foundations, and on the contact of geometry with philosophy on the one hand, and with the physical world on the other. A paragraph by Hessenberg, which illustrates this, would serve even better to elucidate Reidemeister's work than his own. "Only that part of geometry should be called 'coordinate geometry' which numerically fixes objects that can be geometrically visualized. We have the purest form of coordinate geometry when these numbers are not taken from the sphere of arithmetic, but derived from geometric relations. 'Analytic geometry' is something quite different. In its purest form it avoids reference to visualizable reality; in place of that, it builds an artificially abstract geometry with the numbers of arithmetic."

Except when setting up artificial geometries, Hessenberg gives us coordinate geometry. Reidemeister, in his remarkably well planned work, has a section devoted to each type. The care with which the design was made is indicated, not only by the contents, but also by the omissions. Thus we miss, of the topics treated in the other book, angles, superficial and spatial measurement; and we should probably never hear of a third dimension, were it not for its connection with Pascal's theorem. Points and lines in a plane are all,—but it is often a more general plane than Euclid could have imagined. There is no mention of the classical non-euclidean geometries.

In the introduction,—abstract, simple, general,—the idea of congruence is basic; congruence is the equivalence of elements in any universe under any group of transformations. A second important concept is that of the reference set (Bezugsmenge),—a set of elements (not exhausting the class under consideration) such that every non-identical transformation of the group considered displaces at least one element. Clearly the addition of any number of elements to a reference set yields either the entire class of elements or else another reference set—a fact which might well have been mentioned. Usually Reidemeister speaks only of reference sets of minimum size. Thus, if the elements are the points of a plane (pairs of real numbers), a reference set for the group of motions consists of two points at a fixed distance, that for the affine group of three linearly independent points, and for the projective group of four points, of which no three are linearly dependent.

Only when each element is a set of real numbers is the extent of the reference set left arbitrary, and that is merely in the first definition of "natural coordinates." A particular reference set $\mathfrak{B}_0$, consisting of m n-ples, is chosen. If another reference set $\mathfrak{B}$, together with an element Z corresponds, by a transformation of the group considered, to the set $\mathfrak{B}_0$ and the element Z_0 (which consists of the numbers $(z_1, z_2, \cdots, z_n)$), then these numbers z_i are the natural coordinates of Z with respect to $\mathfrak{B}$. The values of m and n are not restricted.

The second chapter develops algebras from the axioms of operations and of order, so that analytic geometry may be built on logically founded number systems. Reidemeister considers not only the number systems obeying all the usual axioms of operations—and therefore forming fields—but also, and with greater interest, those in which one or another axiom does not hold. By analogy with Hessenberg's language, we might speak of "artificial algebras." Espec'al emphasis is laid on non-commutative multiplication, but we have also non-archimedean ordered fields and unilaterally distributive number systems.

There is one part of this chapter which is not quite satisfactory—a section entitled "Real numbers as ordered groups," which leads the way to the arithmetization of vectors in Part II. Its object is to show that the real numbers constitute but a special case of systems satisfying the following requirements:

1. The numbers a, b, $\cdots$ with the combination $a+b$ form an ordered group. (Let us call it $\mathfrak{G}$.)

2. Every number can be uniquely halved.

3. If a and b are two numbers and if, for every rational r, $r>a$ and $r>b$ are either simultaneously true or simultaneously false, then $a=b$. (This is equivalent to Archimedes' axiom.)

Now bisection yields, not all rational numbers, but only those of form $k/2^n$. Accordingly, the assumption that every cut in these dual fractions can correspond to but one real number is not, without further analysis, justified by Axiom 3.

Once it is demanded that to every cut there correspond a number of $\mathfrak{G}$, the numbers of the class are the products of the real numbers and of a. Each automorphism of their group has the form $\xi=\xi\iota$, ι being a real number. Reidemeister then states that the class consists of the real numbers themselves only if $\iota=1$. This is correct, but the true reason seems not to be mentioned. There is no provision for multiplication within $\mathfrak{G}$, and therefore no unit element. Were there a unit and multiplication, then $\mathfrak{G}$ would be isomorphic with the system of real numbers.

Chapter 3 develops affine and projective geometry. The most notable feature is the omission of the axiom of commutative multiplication. Of course the algebra must be developed without determinants, and it is done elegantly. A weakness for the word "umgekehrt" is noted. If the hypothesis of a theorem has five items, of course there are five possible converses. In Chapter 5, Reidemeister speaks of "the further converses of Theorem 1". What, then, can we make of the last sentence of this theorem? "Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$, $\bar{\mathfrak{a}}_1$ and $\bar{\mathfrak{a}}_2$ be two pairs of parallel vectors; if, then, $\mathfrak{a}_2=\mathfrak{a}_1 a$, $\bar{\mathfrak{a}}_2=\bar{\mathfrak{a}}_1\bar{a}$ and $a=\bar{a}$, then the vectors $\mathfrak{a}_1-\bar{\mathfrak{a}}_1$ and $\mathfrak{a}_2-\bar{\mathfrak{a}}_2$ are parallel. The converse is also true, provided $\mathfrak{a}_1$ and $\bar{\mathfrak{a}}_1$ are not parallel."

The second part of the book, embodying results published in the last four years by Reidemeister and others, builds up geometry on non-algebraic axioms. The connection with analytic geometry is then established through geometric entities which correspond to numbers. For this purpose there is a separate axiom equivalent to each axiom on algebraic operations; and every such axiom is a statement of the closure of a particular figure—of the incidence of a point and a line in a configuration. For instance, Pascal's theorem holds if and only if multiplication is commutative. Commutativity of addition is given by the following axiom (2)—in which $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ lines are, respectively, members of three coplanar parallel pencils— "If P_1, P_2; Q_1, R_1; Q_2, R_2 lie on three $\mathfrak{A}$ lines; R_1, R_2; Q_2, P_2; Q_1, P_1 on three $\mathfrak{B}$ lines; Q_1, Q_2; P_2, R_1 on two $\mathfrak{C}$ lines, then R_2, P_1 also lie on a $\mathfrak{C}$ line" (p. 55).

This system of lines, called a 3-web (3-Gewebe) is the frugal equipment with which the first axioms, and a wealth of subsequent developments, are concerned. Incidentally, Reidemeister briefly discusses an unusual type of "improper points." The composition of vectors is indicated by multiplication, homogeneous coordinates replace non-homogeneous ones, and the points on the fundamental triangle are named improper. The three pencils are now determined by the triangle's vertices, and thus the proper points of the 3-web are mapped on the interior of the triangle. (There is similarity to the interior of a triangle inscribed to the absolute conic in hyperbolic geometry, but it is only the straight lines through the vertices in one case which obtain linear equations in the other.)

The addition of a fourth pencil $\mathfrak{D}$, with a proper point as vertex, and of lines obtained from these by translations in the $\mathfrak{A}$ direction, completes the material for a plane. Simple projective operations define equality and proportionality, addition and multiplication, so that each ratio of collinear vectors can be called a number. The noteworthy feature of this development is the geometric basis for number systems which need not obey all the usual operational axioms. If all of these, except perhaps commutativity of multiplication, are satisfied, the plane is affine.

The final chapter is devoted to projective geometry, the transition from affine geometry having again as chief novelty non-commutative multiplication. Thus, the fundamental theorem takes the form of this statement on cross ratios. "If γ and γ^* are two projectively related lines, and U, O, E, V correspond to U^*, O^*, E^*, V^*, then there is such a number v_0 that

$$v(U^*, O^*, E^*, V^*) \equiv v_0 v(U, O, E, V) v_0^{-1}."$$

Pascal's theorem then brings us to the usual geometry. The book closes with a clear discussion of the relation between its two parts, of the need for analytic geometry, and with a statement of problems as yet unsolved.

A few changes would make the work easier reading. If an automorphism were always given that name, there would be a gain. In the statement of unusual algebras there should be greater care. Thus, on page 38, line 4, ρ^2 should be replaced by ρ. On page 41, the sum of $a=(\alpha_1, \alpha_2)$ and $b=(\beta_1, \beta_2)$ is of course not $(\alpha_1+\alpha_1, \beta_1+\beta_2)$, but $(\alpha_1+\alpha_2, \beta_1+\beta_2)$.

Excessive brevity, especially toward the end, is hard on the reader. Equality of $\mathfrak{A}$ vectors in an $\mathfrak{ABC}$ web (Vektorgleichheit) and in an $\mathfrak{ABD}$ web (Masszahlengleichheit) do not mean the same thing, yet they are both called equality; this is especially confusing when, as on page 117, all four pencils are named. And, finally, it is to be hoped that the author will supply the next edition with an index and with an appendix containing all the axioms, to which reference must frequently be made.

Both books are excellently printed. Springer draws attention to his clear, neat page by a footnote on page 130 in contrasting type.

A good geometrical library will certainly wish to possess both works, the more so since they so surprisingly avoid duplication of subject matter. If a reader studies first the Hessenberg work, with its appeal to the pictorial sense, and then the sterner, very stimulating book of Reidemeister, he can gain greatly in his knowledge of the logic of geometry.

E. S. ALLEN

FOUR FRENCH BOOKS ON HYDRODYNAMICS

(1). *Leçons sur la Hydrodynamique.* By H. Villat. Paris, Gauthier-Villars, 1929. iii+296 pp.

(2). *Leçons sur la Théorie des Tourbillons.* By H. Villat. Paris, Gauthier-Villars, 1930. ii+300 pp.

(3). *Mécanique des Fluides.* By H. Villat. Paris, Gauthier-Villars, 1930. vii+175 pp.

(4). *Leçons sur la Résistance des Fluides non Visqueux.* By P. Painlevé. Première partie rédigée par A. Metral. Paris, Gauthier-Villars, 1930. iv+183 pp.

Within the last decade or two the interest in research in hydrodynamics has spread widely throughout all European countries, almost without exception, and very important progress is being achieved in various branches of theoretical and applied hydrodynamics. It is the more striking and disappointing that America has practically no share in this recent progress of hydrodynamics, at least as far as theory is concerned. The amount of papers on theoretical hydrodynamics published in this country is practically negligible. This is deplorable not only because of the high intrinsic interest and practical value of hydrodynamics, but also because in hydrodynamics we have an excellent field of application and means of sharpening of various mathematical tools. Suffice it to say that the recent progress of hydrodynamics mentioned above is intimately related to the modern developments of the theory of functions of a complex variable, conformal mapping, potential theory, integral and integro-differential equations and the like, being in many cases not only the after-effect but even the cause of the development. The last but not the least important aspect of hydrodynamics is the pedagogical one: there is perhaps no better way of lliustrating abstract mathematical theories and of developing in students a critical ability in applying and interpreting mathematical results. It is to be hoped that the recent books published in Germany (by Oseen and Lichtenstein)