# ANALYTIC FUNCTIONS AND MATHEMATICAL PHYSICS*

BY G. Y. RAINICH

1. *Some Properties of Analytic Functions.* We begin by reviewing briefly some fundamental points in the theory of analytic functions in a form which will be convenient for further references. Departing slightly from customary notations, we shall write $w=v+iu$, and we shall consider the theory primarily as the theory of a system of differential equations

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

which are called the Cauchy-Riemann equations.

We shall not enter into various fine points which arise in the discussion, but we may mention that the functions $u$ and $v$, as well as the other functions which appear later, must be assumed to be differentiable, that is, to possess complete differentials in the sense of Stolz.

(a) One point of view often taken in applications is that we have a vector or, rather a vector field, of components $(u, v)$, and that the differential equations express the fact that the rotation (curl) and the divergence of this vector are zero.

Another point of view, which we shall find extremely useful, is that we have *two* vectors $f$ and $r$, whose components are $f_1=u$, $f_2=v$, and $r_1=v$, $r_2=-u$, respectively, and that the differential equations express the fact that the divergences of both are zero. These two vectors, as the relations

$$f_1=-r_2, \qquad f_2=r_1, \tag{2}$$

show, are perpendicular and of equal length, so that we may say that the theory of analytic functions is the theory of two equal and perpendicular vectors in the plane, with zero divergences.

(b) The differential equations may be considered as integrability conditions. They are equivalent to the vanishing of certain

* This paper covers in part the subject matter of a symposium lecture given by the author before the Society in Chicago, April 3, 1931.

contour integrals, taken around contours inside of which the functions are regular.

(c) If the functions are not regular inside a contour but have singularities in one point, the integrals do not in general vanish, but furnish two real numbers (which taken together as a complex number form a residue) and which characterize to a certain extent the singularity. The simplest type of singularity is shown by the function $w=ci/z$, for which in our notations we have

$$u = \frac{cx}{r^2}, \qquad v = \frac{cy}{r^2}; \tag{3}$$

the residue in this case is equal to $ci$. (Here $c$ is a real number and $r^2=x^2+y^2$.)

(d) By elimination of one of the functions $(u, v)$, we obtain for the other the second-order (Laplace) equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{4}$$

(e) There are certain second-degree quantities that might be discussed in connection with an analytic function; these are

$$W = \tfrac{1}{2}w^2 = \tfrac{1}{2}(v^2 - u^2) + iuv, \tag{5}$$

half the square of the function; and

$$Q = w\cdot\bar{w} = u^2 + v^2 = \tfrac{1}{2}(f_1^2 + f_2^2 + r_1^2 + r_2^2), \tag{6}$$

the square of the modulus, or the norm of the function. By the fact that $(u, v)$ satisfy the differential equations (1) certain conditions are imposed on these second-degree quantities, and we shall now write them down.

For $W$ it is easy; the statement that $w$ is an analytic function means the same as the statement that $\frac{1}{2}w^2$ is an analytic function, and the latter statement is equivalent to the relations

$$\begin{cases} \dfrac{\partial \frac{1}{2}(u^2 - v^2)}{\partial x} + \dfrac{\partial uv}{\partial y} = 0, \\[2ex] \dfrac{\partial uv}{\partial x} + \dfrac{\partial \frac{1}{2}(v^2 - u^2)}{\partial y} = 0. \end{cases} \tag{7}$$

This system of differential equations is then equivalent to system (1), and it is easy to verify this fact directly.

The situation is more complicated with respect to $Q$. The easiest way to arrive at the differential equation which must be satisfied by $Q$ as a result of the fact that $u$, $v$ satisfy (1) seems to be the consideration of $\log w^2$. This must be analytic if $w$ is, and therefore the real part of $\log w^2$ must satisfy the Laplace equation. But the real part of $\log w^2$ is $\log w\bar{w}$, so that we have

$$\frac{\partial^2 \log Q}{\partial x^2} + \frac{\partial^2 \log Q}{\partial y^2} = 0$$

or

$$Q\cdot\left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2}\right) = \left(\frac{\partial Q}{\partial x}\right)^2 + \left(\frac{\partial Q}{\partial y}\right)^2. \tag{8}$$

It is interesting to note that this equation is non-linear, and that it therefore gives an example of a non-linear equation which is a consequence of a system of linear equations. We next ask ourselves whether the condition expressed by the last equation is sufficient, in other words whether every function $Q$ satisfying this condition may be considered as the norm of an analytic function.

In the first place it is clear that a function is not entirely determined by its norm. If the function $w=v+iu$ has the norm $Q$, any function $V+iU$, for which

$$U = u\cos\phi - v\sin\phi,\quad V = u\sin\phi + v\cos\phi, \tag{9}$$

where $\phi$ is an arbitrary function of $x$ and $y$, will have the same norm. The question then reduces to this: are there among the functions $V+iU$ some that are analytic? An easy calculation leads to the result that the equations

$$\begin{aligned} &\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + u\cdot\phi_1 + v\cdot\phi_2 = 0, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - v\cdot\phi_1 + u\cdot\phi_2 = 0, \end{aligned} \tag{10}$$

with $\phi_1=\partial\phi/\partial x$, $\phi_2=\partial\phi/\partial y$, must be satisfied. Further calculation shows that the integrability condition for these equations is exactly the above equation (7) for $Q$.

(f) An analytic function can be developed into a power series. In view of an analogy that we want to establish later the following way to introduce this series might find its place here. We first write the Cauchy-Riemann equations (1) in the form $\partial w/\partial x + i\partial w/\partial y = 0$, and, passing to polar coordinates, we have

$$r\frac{\partial w}{\partial r} + i\frac{\partial w}{\partial \theta} = 0. \tag{11}$$

Next, making use of the fact that the equation does not contain $\theta$ explicitly, and therefore must allow solutions of the form

$$Pe^{ik\theta}, \tag{12}$$

where $P$ is independent of $\theta$, we find for $P$ the equation

$$r\frac{\partial P}{\partial r} - kP = 0,$$

which has a solution $P = r^k$, so that

$$w = r^k e^{ik\theta} = z^k \tag{13}$$

is a solution of the original equation. If we require the solution to be one-valued, $k$ must be an *integer*, and if we want it to be continuous (at the origin), $k$ must be *non-negative*. A linear combination of a finite number of such solutions is a polynomial, and the general solution may be presented as a linear combination of an infinite number of such solutions which we may consider as the limit of a (uniformly convergent) sequence of polynomials, or, if you prefer, as a power series.

2. *The Volterra Theory*. We pass now to generalizations. We saw that the theory of analytic functions may be considered as the theory of two equal and perpendicular vectors with vanishing divergences. We may try to extend this to the three-dimensional space. If we take two vectors $f$ and $r$ and write down the conditions for their equality and perpendicularity, we find

$$f_1^2 + f_2^2 + f_3^2 = r_1^2 + r_2^2 + r_3^2, \quad f_1 \cdot r_1 + f_2 \cdot r_2 + f_3 \cdot r_3 = 0 \tag{14}$$

and although the differential equations

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0, \quad \frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial y} + \frac{\partial r_3}{\partial z} = 0 \tag{15}$$

are linear, the system as a whole is *not*. The same is true in the two-dimensional case, of course, but there the equations are reduced to linear equations easily. We shall not consider now the question whether these equations can be reduced to linear equations (see §6); trying to keep more or less to the historical order we shall outline a way out from this difficulty, which leads to Volterra's theory of conjugate functions dated back to 1889.*

We may consider a vector as a finite portion of a directed line (or curve), and its components as the lengths of the projections of this finite portion on the coordinate axes. If now we consider a finite portion of a plane, or surface, as a *surface vector*, and as its components the *areas* of its projections on the coordinate planes, we have a new object to operate upon. The general case may be reduced to that of a triangle with vertices at (0, 0, 0), $(x_1, x_2, x_3)$, $(y_1, y_2, y_3)$. The areas of the projections are the determinants of the matrix

$$\frac{1}{2}\left\|\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right\|,$$

which we may denote by $R_{23}$, $R_{31}$, $R_{12}$ or $L$, $M$, $N$, or in general

$$R_{ij} = \frac{1}{2}\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}.$$

It may seem sufficient to consider only the above three components; however, it is more convenient not to restrict $i$ and $j$, but to use all nine combinations, introducing the relation $X_{ij}+X_{ji}=0$. These nine may be arranged into a square matrix

$$(16) \qquad \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}, \quad \begin{pmatrix} O & N & -M \\ -N & O & L \\ M & -L & O \end{pmatrix},$$

so that a surface vector is represented by a matrix possessing the property of antisymmetry.

Consider now a *line* vector $X=F_1$, $Y=F_2$, $Z=F_3$ with the above *surface* vector. Conditions of perpendicularity and nu-

* V. Volterra, *Sulle funzioni conjugate*, Rendiconti dei Lincei, (4), vol. 5 (1889), pp. 599–611. See also Rendiconti di Palermo, vol. 3 (1889), pp. 260–272.

merical equality may be shown to result simply in the equations

$$(17) \qquad R_{23} = F_1, \quad R_{31} = F_2, \quad R_{12} = F_3,$$

which should be compared with the equations (2). If now we impose the vanishing of divergences

$$\sum \frac{\partial F_\alpha}{\partial x_\alpha} = 0, \quad \sum \frac{\partial R_{i\alpha}}{\partial x_\alpha} = 0,$$

and write out everything without indices in terms of $X$, $Y$, $Z$, $x$, $y$, $z$, we obtain a system of four equations

$$(18) \quad \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0, \quad \frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}, \quad \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}.$$

This is an analog for three-space of the Cauchy-Riemann system. It is worth noting that setting $Z=0$ we get two functions $X$ and $Y$ depending on $x$ and $y$ alone which satisfy the equations

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0, \qquad \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y},$$

that is, exactly the Cauchy-Riemann equations for $X=u$, $Y=v$. We further note that the elimination of two of the three functions $X$, $Y$, $Z$ from the above four equations leads as in (d), §1, to the Laplace equation in three dimensions. We have thus a theory in three-space which may be considered as a generalization of the theory of analytic functions, and which is essentially the theory of two equal and perpendicular vectors with vanishing divergences, one of these vectors being a line vector, and the other a surface vector.

There are no essential difficulties (except that of losing the help of intuition) in extending the theory to any number of dimensions. We shall only take up the case $n=4$. Here we may consider the case of two perpendicular *surface* vectors with vanishing divergences. There are here *six* coordinate planes, and therefore every surface vector has six components. They are known as *six-vectors*. We may write, using notations similar to those used above, $f_{ij}$ for the components, with the condition $f_{ij}+f_{ji}=0$. In four-space, this will mean a square four-rowed matrix, whose sixteen elements reduce to six as a result of these relations. The conditions of numerical equality and perpendicularity for two such vectors $f$ and $r$ again reduce to linear relations, namely,

$$(19)\quad f_{41} = r_{23},\ f_{42} = r_{31},\ f_{43} = r_{12},\ f_{23} = r_{41},\ f_{31} = r_{42},\ f_{12} = r_{43},$$

which should be compared with (2) and (18). The divergence equations together with these relations constitute a linear system which is analogous to, and a generalization of, the Cauchy-Riemann system (1). This system may be written as

$$(20)\qquad \sum\frac{\partial f_{i\alpha}}{\partial x_\alpha} = 0,\quad \sum\frac{\partial r_{i\alpha}}{\partial x_\alpha} = 0,$$

or, in full, if we set

$$(21)\quad f_{14} = X,\ f_{24} = Y,\ f_{34} = Z,\ f_{23} = L,\ f_{31} = M,\ f_{12} = N,$$

$$(22)\qquad \left\{\begin{array}{ll} \dfrac{\partial Z}{\partial y} - \dfrac{\partial Y}{\partial z} + \dfrac{\partial L}{\partial t} = 0, & \dfrac{\partial N}{\partial y} - \dfrac{\partial M}{\partial z} + \dfrac{\partial X}{\partial t} = 0, \\[2ex] \dfrac{\partial X}{\partial z} - \dfrac{\partial Z}{\partial x} + \dfrac{\partial M}{\partial t} = 0, & \dfrac{\partial L}{\partial z} - \dfrac{\partial N}{\partial x} + \dfrac{\partial Y}{\partial t} = 0, \\[2ex] \dfrac{\partial Y}{\partial x} - \dfrac{\partial X}{\partial y} + \dfrac{\partial N}{\partial t} = 0, & \dfrac{\partial M}{\partial x} - \dfrac{\partial L}{\partial y} + \dfrac{\partial Z}{\partial t} = 0, \\[2ex] \dfrac{\partial L}{\partial x} + \dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial z} = 0, & \dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z} = 0. \end{array}\right.$$

Here again the elimination of all but one component leads to a Laplace equation.

As I have said, analogous considerations apply to spaces of any number of dimensions, and when the sum of the numbers of dimensions of the two vectors is equal to the number of dimensions of the space the system reduces to a linear system—this is the case studied by Volterra.* Volterra's work has been followed up by Lagally, de Donder, Dixon,† and a few others, but very much remains to be done. Many features of the theory of analytic functions are preserved, but not all. Complex numbers or

---

* Volterra, loc. cit.

† M. Lagally, Münchener Berichte, 1917. Th. de Donder, Bulletin des Sciences de Belgique, 1906, pp. 400–409. A. C. Dixon, Quarterly Journal, vol. 35 (1904), pp. 283–296.

hypercomplex numbers are not used except in the four-dimensional case by Dixon. The three-dimensional theory bears the same relationship to the Newtonian potential as the theory of analytic functions to the logarithmic potential, and some results of potential theory can be directly translated into this theory, but there are some questions which have to be treated independently. Of course, the differential equations may be replaced by integral conditions, by the vanishing of certain integrals taken over surfaces surrounding volumes in which the functions are regular. An extension of the Cauchy integral formula can be proved for all cases. There are different types of singularities, point singularities, line singularities, etc. The theory of residues presents a particular fascination. Expansions analogous to power series exist; in the three-dimensional case they are essentially developments into series of harmonic functions. But we shall abandon now the purely mathematical developments and pass to applications.

3. *Analogous Situations in Physics.* We shall not go farther back than Newton in tracing the development in mathematical physics. With the name of Newton we associate a great many things, but here we are interested in particular in two. First, in the general laws of mechanics stating that the time rate of change of the momentum is equal to force

$$\frac{dmu}{dt} = X, \quad \frac{dmv}{dt} = Y, \quad \frac{dmw}{dt} = Z; \tag{23}$$

and, second, in the special case (inverse square law) in which the force components are given by

$$X = \frac{cx}{r^3}, \quad Y = \frac{cy}{r^3}, \quad Z = \frac{cz}{r^3}, \tag{24}$$

which should be compared with (3). We have here the attracting point, and the attracted point, both as discrete points. One of the trends of physics has been the transition from the consideration of discrete points to the consideration of fields, that is, continuously distributed quantities. We shall follow this trend separately for the right-hand sides and the left-hand sides of the equations, that is, the forces and matter.

First, as to the forces, we find easily that the force components given by the explicit formulas (24) satisfy the differential equations

$$(25)\quad \frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0,\quad \frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x},\quad \frac{\partial X}{\partial z}=\frac{\partial Z}{\partial x},\quad \frac{\partial Y}{\partial z}=\frac{\partial Z}{\partial y}.$$

This is at the same time a generalization of the inverse square law and its source. We can get the inverse square law back from these differential equations by solving them under the additional condition of symmetry around a point. In passing from an explicit expression for the law of force to the partial differential equations we shifted our attention, without changing any essential features of the situation, from the singularities of the field to the field connected with these singularities.

We may do a similar thing to the left-hand sides of our equations, and introduce a continuous fluid instead of a discrete particle (the case of a particle may then be obtained as a limiting case of infinite density). Mathematically, this transition is a transition from ordinary to partial differential equations. In the simplest case, that of steady motion, when density is constant, and there exists a potential of velocities, the equations satisfied by the components of velocity are

$$(26)\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x},\quad \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x},\quad \frac{\partial v}{\partial z}=\frac{\partial w}{\partial y}.$$

These equations were known in the 18th century; they appear for instance, in a memoir of Lagrange* in 1760; the remarkable thing is that these equations of the simplest motion of a fluid are the same as those of the simplest field of force. Furthermore these equations are exactly the case $n=3$ of the Volterra theory (§2) and thus a generalization of the equations (1) of the theory of analytic functions. The inverse square law appears as the simplest singularity corresponding to $w=c/z$ (compare formulas (3) and (24)); the exponent in the denominator is equal to the number of dimensions in each case. The residue corresponds to the mass, or electric charge of the attracting point.

In considering cases more general than the simple one just mentioned the ways of the forces and those of the velocities

* Lagrange, Oeuvres, vol. 1, p. 442.

part, at least for a time. For hydrodynamics we have to use in the general case the equations of Euler, which we take in the following form:

$$
(27)\qquad \left\{\begin{aligned}
&\frac{\partial\rho u^2}{\partial x}+\frac{\partial\rho uv}{\partial y}+\frac{\partial\rho uw}{\partial z}+\frac{\partial\rho u}{\partial t}=X+\frac{\partial p}{\partial x},\\
&\frac{\partial\rho vu}{\partial x}+\frac{\partial\rho v^2}{\partial y}+\frac{\partial\rho vw}{\partial z}+\frac{\partial\rho v}{\partial t}=Y+\frac{\partial p}{\partial y},\\
&\frac{\partial\rho wu}{\partial x}+\frac{\partial\rho wv}{\partial y}+\frac{\partial\rho w^2}{\partial z}+\frac{\partial\rho w}{\partial t}=Z+\frac{\partial p}{\partial z},\\
&\qquad\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}+\frac{\partial\rho w}{\partial z}+\frac{\partial\rho}{\partial t}=0,
\end{aligned}\right.
$$

where $p$ is the pressure and $\rho$ is the density. The first three of these equations correspond to the three equations (23), and are nothing but their translation into the language of continuous distribution; the last one is a translation of the equation $dm/dt=0$, which is usually not written out explicitly in the discrete case.

The first thing we notice about these equations is that they deal not with the quantities $u$, $v$, and $w$, but with quadratic expressions in them. In the simplest case $n=2$, replacing the pressure $p$ by $\frac{1}{2}(u^2+v^2)$ (internal pressure), the density by 1, and the forces by zero, we get the equations

$$
(28)\qquad \frac{\partial\frac{1}{2}(u^2-v^2)}{\partial x}+\frac{\partial uv}{\partial y}=0,\quad \frac{\partial uv}{\partial x}+\frac{\partial\frac{1}{2}(v^2-u^2)}{\partial y}=0,
$$

which are exactly the equations (7) for what we have called $W$ in the theory of analytic functions. With the same assumptions, but for $n=3$, we get

$$
(29)\qquad \left\{\begin{aligned}
&\frac{\partial\frac{1}{2}(u^2-v^2-w^2)}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}=0,\\
&\frac{\partial vu}{\partial x}+\frac{\partial\frac{1}{2}(-u^2+v^2-w^2)}{\partial y}+\frac{\partial vw}{\partial z}=0,\\
&\frac{\partial wu}{\partial x}+\frac{\partial wv}{\partial y}+\frac{\partial\frac{1}{2}(-u^2-v^2+w^2)}{\partial z}=0.
\end{aligned}\right.
$$

We naturally ask about the relation between these differential equations for the quadratic quantities and the differential equations written out before for the linear quantities. As in the case $n=2$, the equations for the second-degree quantities are consequences of those for the linear quantities, but in contradistinction from that case we cannot pass here from (29) to (26); this raises the question: what additional conditions have to be imposed on the quadratic quantities so that the linear equations would be satisfied?

The form of Euler's set of equations suggests the consideration of the case $n=4$, by considering time as the fourth coordinate and $t$ as the fourth component of the vector $(u, v, w, t)$. Taking this suggestion seriously means making the first step toward the special theory of relativity. Without entering into details we must mention that velocity is represented in this theory by a four-vector, whose components we may denote by $u_i$, and that if we consider from this point of view matter in the absence of forces, we arrive at a set of equations which constitute the generalization for four dimensions of (28) and (29).

We shall now return to the consideration of forces. We spoke of the inverse square law without mentioning the *physical nature* of the forces. Historically, the gravitational field was the one for which in the time of Newton the inverse square law was formulated. But we are now more interested in the electrostatic and magnetostatic fields to which the same law has been applied in the nineteenth century. To each of these fields taken separately that law, and therefore the equation (25), applies. As soon as the time is introduced, however, that is, as soon as we begin to consider fields changing with time, interaction between these fields appears, and we have to consider them together. The result may be formulated as follows. In the static case the quantities $\partial Y/\partial z-\partial Z/\partial y$, $\partial Z/\partial x-\partial X/\partial z$, $\partial X/\partial y-\partial Y/\partial x$ and $\partial M/\partial z-\partial N/\partial y$, $\partial N/\partial x-\partial L/\partial z$, $\partial L/\partial y-\partial M/\partial x$ were zero. For the case when the fields change with time, Faraday's discoveries coupled with Maxwell's imagination resulted in equating these quantities (I omit the factor of proportionality), respectively, to $\partial L/\partial t$, $\partial M/\partial t$, $\partial N/\partial t$ and $-\partial X/\partial t$, $-\partial Y/\partial t$, $-\partial Z/\partial t$, so that together with the equations expressing the vanishing of the divergences we have the following set of equations for electromagnetic forces in matter-free space (*Maxwell's equations*):

$$
(30)\quad \begin{cases}
\dfrac{\partial Y}{\partial z}-\dfrac{\partial Z}{\partial y}-\dfrac{\partial L}{\partial t}=0, & \dfrac{\partial M}{\partial z}-\dfrac{\partial N}{\partial y}+\dfrac{\partial X}{\partial t}=0,\\[2mm]
\dfrac{\partial Z}{\partial x}-\dfrac{\partial X}{\partial z}-\dfrac{\partial M}{\partial t}=0, & \dfrac{\partial N}{\partial x}-\dfrac{\partial L}{\partial z}+\dfrac{\partial Y}{\partial t}=0,\\[2mm]
\dfrac{\partial X}{\partial y}-\dfrac{\partial Y}{\partial x}-\dfrac{\partial N}{\partial t}=0, & \dfrac{\partial L}{\partial y}-\dfrac{\partial M}{\partial x}+\dfrac{\partial Z}{\partial t}=0,\\[2mm]
\dfrac{\partial L}{\partial x}+\dfrac{\partial M}{\partial y}+\dfrac{\partial N}{\partial z}=0, & \dfrac{\partial X}{\partial x}+\dfrac{\partial Y}{\partial y}+\dfrac{\partial Z}{\partial z}=0.
\end{cases}
$$

It seems surprising now that until 1907 it was not noticed that the best way to write these equations was the four-dimensional way already suggested by the form of Euler's equations (27). The explanation is that the equations were encumbered with matter and three-dimensional vector analysis. Moreover, the really fundamental things have a way of appearing to be simple once they have been stated by a genius, who was in this case Minkowski.

Another thing is almost as remarkable—the equations at which we have arrived are, except for some differences in sign, the same as the equations (20) of the Volterra theory for $n=4$, $r=2$. It remained for L. Hanni* to notice this fact in 1910. We have thus again in mathematical physics a generalization of the theory of analytic functions. The difference in sign is of extreme importance in physics. It may be stated that without this difference our world would have been dark; the minus sign that appears in the above quantities makes light possible. In this connection we might mention that if we eliminate all dependent functions but one from Maxwell's equations, we obtain, due to this difference in sign, instead of the Laplace equation that appears as a result of such elimination in the cases considered above, an equation of the type

$$
(31)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=0,
$$

* L. Hanni, *Ueber den Zusammenhang zwischen den Cauchy-Riemannschen und den Maxwellschen Differentialgleichungen*, Tôhoku Mathematical Journal, vol. 5 (1910), pp. 142–175.

which differs from the Laplace equation by the minus sign before the last term. This is the so called *wave equation*; it received its name from the fact that it is often used in describing wave phenomena, especially in acoustics, and in optics in connection with what is called the wave theory of light. From the mathematical point of view the introduction by Maxwell of the electromagnetic theory of light may be considered as essentially the substitution for a theory based on one second-order partial differential equation of a theory based on a system of first-order equations (of which the second-order equation is a consequence). We want to emphasize this fact in view of an analogous situation to be mentioned later. We may mention also the fact that the main importance of this substitution seems to lie in its effect in introducing boundary conditions.

The lack of beauty due to the difference in sign may be remedied by the introduction of slight modifications in our notations. If we use instead of (21), the notations

$$(32)\quad \begin{cases} x = x_1, & y = x_2, & z = x_3, & t = ix_4, \\ X = if_{41}, & Y = if_{42}, & Z = if_{43}, & \\ L = f_{23}, & M = f_{31}, & N = f_{12}, & \end{cases}$$

and instead of (19) the relations

$$(33)\quad ir_{41} = f_{23},\ ir_{42} = f_{31},\ ir_{43} = f_{12},\ ir_{23} = f_{41},\ ir_{31} = f_{42},\ ir_{12} = f_{41},$$

the equations of Maxwell assume exactly the form (20). The introduction of these notations constitutes the second step toward the special theory of relativity.

4. *Second-Degree Quantities in Electrodynamics.* We noticed before that in Euler's hydrodynamic equations (27) the quantities $u$, $v$, $w$ appear not directly, but as certain quadratic combinations; in the two- and three-dimensional cases written out above (28) and (29) the quantities subjected to differentiation are the elements of the matrices

$$(34)\quad \begin{pmatrix} \frac{1}{2}(u^2 - v^2) & uv \\ uv & \frac{1}{2}(v^2 - u^2) \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} \frac{1}{2}(u^2 - v^2 - w^2) & uv & uw \\ vu & \frac{1}{2}(v^2 - u^2 - w^2) & vw \\ wu & wv & \frac{1}{2}(w^2 - u^2 - v^2) \end{pmatrix}.$$

The differential equations (28) and (29) may be expressed by stating that the divergences of these matrices vanish. In four-space (special relativity theory) there also appears a matrix of the same nature which we write out in index notation:

$$M_{ij} = \rho(u_i u_j - \tfrac{1}{2}\delta_{ij}u_\alpha u_\beta). \tag{35}$$

In the electromagnetic theory analogous combinations of force components were introduced by Maxwell under the name of stress components. When we treat, in three-dimensional space, electric and magnetic forces separately, the stresses are

$$\begin{array}{ccc|ccc} \multicolumn{3}{c}{\text{Electric Stresses}} & \multicolumn{3}{c}{\text{Magnetic Stresses}} \\ X^2 - e & XY & XZ & L^2 - m & LM & LN \\ YX & Y^2 - e & YZ & ML & M^2 - m & MN \\ ZX & ZY & Z^2 - e & NL & NM & N^2 - m \end{array} \tag{36}$$

with $e = X^2 + Y^2 + Z^2$; with $m = L^2 + M^2 + N^2$.

We may note that if we impose on the forces the equations (25), the divergence of the stress matrix vanishes, but we shall return to the question of differential relations later. We saw that the electric and magnetic forces are interrelated, and that the situation is best expressed in four-dimensional notations. Therefore there seems to be no reason for keeping the electric and magnetic stresses apart, and so we combine the above matrices by simply adding the corresponding elements together. We extend the three-rowed matrix thus obtained into a four-rowed matrix by adding a fourth row and a fourth column. We shall not write out this four-rowed matrix in full, nor shall we explain the physical meaning (which is a very important one) of the new additional quantities. Instead we shall write it in short notations which will bring out even more clearly the relation to the theory of analytic functions. We have here two surface vectors $f$ and $r$ represented by four-rowed matrices as our first-degree quantities. In terms of them, the four-rowed matrix of the stresses is

$$E = \tfrac{1}{2}(f^2 + r^2), \tag{37}$$

or, in index notation,

$$E_{ij} = \tfrac{1}{2} \sum (f_{i\alpha} f_{\alpha j} + r_{i\alpha} r_{\alpha j}). \tag{38}$$

Here $f^2$ and $r^2$ may be considered as matrices obtained from the matrices $f$ and $r$ by squaring them in the manner employed in squaring a determinant. The expression obtained is a matrix which depends quadratically on the force matrix. The relation between the force matrices $f$ and $r$ on the one hand, and the matrix $E$ on the other is analogous to the relation between the first-degree and the second-degree quantities in the theory of analytic functions. In some ways it reminds us of the quantity $W$. It is, like $W$, a matrix, and the analogy between $W$ and $E$ guided us in the formation of $E$, especially in the three-dimensional stage. As a finished product, however, $E$ resembles more $Q$. In the first place, it possesses the same form (compare formulas (6)); in the second place, the relationship between $E$ and $Q$ appears very clearly when we ask ourselves whether the forces given by $f$ are determined when $E$ is given. Just as in the case of analytic functions, if $(X, Y, Z)$ and $(L, M, N)$ produce certain stresses, the quantities

$$(39)\left\{\begin{array}{l} X\cos\phi - L\sin\phi,\quad Y\cos\phi - M\sin\phi,\quad Z\cos\phi - N\sin\phi, \\ X\sin\phi + L\cos\phi,\quad Y\sin\phi + M\cos\phi,\quad Z\sin\phi + N\cos\phi, \end{array}\right.$$

where $\phi$ is an arbitrary function of $x$, $y$, $z$, and $t$, produce the same stresses. These formulas are analogous to (9).

We pass now to the consideration of differential equations satisfied by the stresses. The matrix $E$, as was the case with $W$ in the case of an analytic function, satisfies, as a result of $(f, r)$ satisfying Maxwell's equations, a set of linear differential equations which may be written in the form

$$(40)\qquad \frac{\partial E_{i\alpha}}{\partial x_\alpha} = 0,$$

and is strictly analogous to equations (7). This analogy appears more clearly when the last equations are written out in components. But these equations do *not* constitute a *sufficient* condition for a stress matrix to be derived from forces which satisfy Maxwell's equations. Subjecting to Maxwell's equations (30) the above expressions (39) for the forces corresponding to a given stress matrix, we obtain a set of eight equations on the derivatives of the heretofore arbitrary angle $\phi$. The vanishing of the divergence of $E$ (see equations (40)) is exactly the set of conditions for algebraic *compatibility* of these equations. In addition to that, however, certain *integrability* conditions must be satisfied. They furnish another non-linear set of differential equations for $E$. We shall not write out these equations,* but for the purpose of further reference we shall put down the equations on the derivatives of the angle $\phi$, which (derivatives) we shall denote by $\phi_1$, $\phi_2$, $\phi_3$, $\phi_4$:

$$
(41)\quad \left\{
\begin{aligned}
&\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} - \frac{\partial L}{\partial t} - N\phi_2 + M\phi_3 - X\phi_4 = 0,\\
&\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} - \frac{\partial M}{\partial t} + N\phi_1 - L\phi_3 - Y\phi_4 = 0,\\
&\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} - \frac{\partial N}{\partial t} - M\phi_1 + L\phi_2 - Z\phi_4 = 0,\\
&\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} - L\phi_1 - M\phi_2 - N\phi_3 = 0,\\
&\frac{\partial M}{\partial z} - \frac{\partial N}{\partial y} + \frac{\partial X}{\partial t} + Z\phi_2 - Y\phi_3 - L\phi_4 = 0,\\
&\frac{\partial N}{\partial x} - \frac{\partial L}{\partial z} + \frac{\partial Y}{\partial t} - Z\phi_1 + X\phi_3 - M\phi_4 = 0,\\
&\frac{\partial L}{\partial x} - \frac{\partial M}{\partial y} + \frac{\partial Z}{\partial t} + Y\phi_1 - X\phi_2 - N\phi_4 = 0,\\
&\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} + X\phi_1 + Y\phi_2 + Z\phi_3 = 0.
\end{aligned}
\right.
$$

These equations are the analogs of (10).

In what precedes we have treated separately the components of the forces and the components of matter corresponding to the right hand sides and the left hand sides of the original equations of motion (23). Although in the description of the further fate of these quantities the theory of analytic functions—our guiding light—ceases to play a leading role, we do not want to abandon forces and matter without mentioning that they become happily reunited by simply adding together the corresponding elements of the matrices $M_{ij}$ (35) and $E_{ij}$ (38) into the elements of a ma-

* See G. Y. Rainich, Transactions of this Society, vol. 27 (1925), p. 129.

trix $T_{ij}$, and that the equations of motion may then be written simply by saying that the divergence of this new matrix is zero. The new equations of motion differ slightly from the old ones but seem to be in keeping with experiments. At this point another of the theories of pure mathematics becomes of extreme importance, namely, the theory of surfaces together with its generalization, the theory of curved spaces (riemannian geometry). A discussion of the interrelation between these two theories* of pure mathematics, and also the application of the second to physics, space does not permit to take up here. It will suffice to say that the identification of the matrix $T_{ij}$ with a tensor appearing in the theory of curved spaces leads to the general relativity theory.†

5. *The Schroedinger Equation.* In its first stages of development, the quantum theory seemed to be very far removed from the continuous theories of which the theory of analytic functions is a model, but further development resulted in a surprising change which we shall sketch here.

We shall restrict our considerations to what the physicists call the hydrogen atom, or, more precisely, the spectrum of a hydrogen atom. We shall only mention the older theory (1913–1925) due to Bohr, which was based on the analogy with the solar system: a nucleus in the center and an electron going around it and (in the last stage of the theory) spinning at the same time. There were two points of difference: in the first place, not all orbits consistent with Newton's law or with Kepler's law, were permitted, but only a discrete set of orbits; in the second place, the different spectral lines were associated not with orbits but each line was associated with two orbits; the conception was that radiation is produced by an electron *jumping* from one orbit to the other.

We have here, just as at the beginning of mechanics, discontinuity in the distribution of matter, but it is accentuated by discontinuity in the set of permissible orbits and discontinuities in motion (jumps). The trend from discontinuity did not appear here until 1926 when the fundamental papers by Schroedinger

* See G. Y. Rainich, *Ueber die Analytischen Funktionen auf einer Minimalflaeche*, Mathematische Annalen, vol. 101 (1929), pp. 386–393.

† These questions were taken up in the Symposium lecture.

were published,* although it must be said that his work was not entirely independent of some ideas of his predecessors. From our point of view it is interesting to note that Schroedinger formulates his idea in the title of his papers as an attempt to interpret a certain set of numbers that appear in analyzing spectra as the set of characteristic values (Eigenwerte) of a *partial* differential equation. (This corresponds to the transition in mechanics from the theory of a moving point to the theory of the motion of a fluid (§3).) This partial differential equation in the simplest case is

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 - Vu + Eu = 0. \tag{42}$$

Here $u$ is a complex-valued field function, $V$ is a given function (in the special case we are considering, the potential of the field of forces in which the electron moves), $E$ is an undetermined constant, and the units have been chosen in such a way as to simplify the coefficients. It is surprising that equations of this type have not been considered in connection with spectra before. In acoustics the equation

$$\Delta u + k^2 u = 0 \tag{43}$$

was used to determine the different states of vibration of a solid. The different characteristic values, that is, the values of $k$ for which there exist solutions of this equation compatible with certain boundary conditions depending on the geometric configuration and the physical properties of the sounding body, were taken to correspond to the different pitches of sound produced. In quantum optics, however, we have instead of boundary conditions the additional condition that the solutions should be one-valued and continuous. This should be compared with the conditions imposed on the solutions of differential equations in the derivation of the power series as a general analytic function in (f), §1. The meaning of the potential $V$ is that by changing it we obtain different sets of characteristic values, that is, different spectra, so that it takes over some functions of the boundary conditions, because in acoustics we ordinarily get different sets of characteristic values by taking different boundary conditions.

* E. Schroedinger, *Quantisierung als Eigenwertproblem*. First paper, Annalen der Physik, vol. 79 (1926), p. 361; second paper, ibid., p. 489; third paper, ibid., vol. 80 (1926), p. 437.

We shall now trace, following Sommerfeld,* the series of changes that the Schroedinger equation underwent, but we shall first call attention to the fact that the connection with the theories discussed heretofore is established by the fact of the appearance of the Laplace differential operator $\Delta u = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. To begin with, time does not appear in the equation, but we may introduce it by reversing the process by which we got rid of $\theta$ in the equation (11). We thus arrive at the equation

$$(44) \qquad \Delta w - Vw + i\cdot(\partial/\partial t)w = 0,$$

which contains differentiation with respect to time but does not contain time explicitly in its coefficients; the substitution

$$(45) \qquad w = ue^{iEt}$$

brings us back exactly to the equation we had before.

This substitution is of the same nature as (12) employed before while we were obtaining a power series as a solution of the Cauchy-Riemann equations. Its justification is here the same (the equation does not contain $t$ explicitly)† and the result is analogous; the general solution of the equation (44) appears as a series whose terms are arbitrary constants multiplied by functions of the type $u_n e^{iE_n t}$, where $u_n$ depend on $x$, $y$, $z$ only, and $E_n$ is a discrete set of constants.

The equation we are considering now is not quite satisfactory from the aesthetic point of view; its form is not in keeping with the principle of relativity which for our present purposes may be formulated as the principle of treating time and space coordinates alike; but it was found that the equation

$$(46) \qquad \begin{aligned} &\left(\frac{\partial}{\partial x} + \phi_1\right)^2 w + \left(\frac{\partial}{\partial y} + \phi_2\right)^2 w + \left(\frac{\partial}{\partial z} + \phi_3\right)^2 w \\ &\qquad - \left(\frac{\partial}{\partial t} + \phi_4\right)^2 w + A^2 w = 0, \end{aligned}$$

* A. Sommerfeld, *Atombau und Spektrallinien*, Wellenmechanischer Ergaenzungsband, Braunschweig, 1929, pp. 119–120.

† This situation is a special case of a more general situation when the equation allows a one-parameter group of transformations (the one-parameter group of transformations being in this case $t' = t + p$); in the general case we also may use a similar device which results in obtaining a sequence of equations involv-

which is more satisfactory in form, gives results which are only slightly different from the results obtained from the preceding equation if we set $\phi_1=\phi_2=\phi_3=0$, $\phi_4=V$, and give to the constant $A$, which is the mass constant, an appropriate value. This improvement of the form without affecting much the numerical results might be compared with the transition from the original Euler equations to the equations mentioned at the end of §4.

We may say that we have now essentially a Laplace-like equation. More precisely, except for the potential $\phi_i$ and the mass term, we have the wave equation. Analogy suggests that we consider two things: second-degree quantities and first-order equations, of which the Laplace-like equation is a consequence.

Beginning with second-order quantities, we notice that $w$ is a complex number and this suggests the consideration of $w\cdot\bar{w}$. If we take for $w$ a particular solution corresponding to one term (13) of the power series in the case of analytic function, namely,

$$w = ue^{iEt},$$

we have $w\cdot\bar{w}=u\cdot\bar{u}$ and we see that $t$ has dropped out. If we take a linear combination of such terms, however, let us say of two such terms,

$$w = a_1u_1e^{iE_1t} + a_2u_2e^{iE_2t}, \tag{47}$$

we get

$$\begin{aligned} w\cdot\bar{w} = {} & a_1\bar{a}_1u_1\bar{u}_1 + a_2\bar{a}_2u_2\bar{u}_2 \\ & + a_1\bar{a}_2u_1\bar{u}_2e^{i(E_1-E_2)t} + a_2\bar{a}_1u_2\bar{u}_1e^{i(E_2-E_1)t}. \end{aligned} \tag{48}$$

The $E$'s are interpreted as energy values, and the set of discrete values of the $E$'s corresponds to the set of orbits in the old Bohr theory. As a result of our forming the second-degree quantity $w\cdot\bar{w}$ we see then that we obtained in a natural way the fact that not the single orbits but pairs of orbits, not the separate energy values, but their differences appear in our formulas; or we may say, the transition from energy values to differences of energy values which in the Bohr theory led to the introduction of jumps, is achieved in the continuous theory by passing from

---

ing one less variable but involving a sequence of numbers, corresponding to the numbers $k$ in the case of an analytic function; these numbers are used in physics as *quantum* numbers.

the linear quantities $w$ to the quadratic quantities $w \cdot \bar{w}$. Another discontinuity is gone. It must be added that there remain apparently great difficulties in the interpretation of the coefficients $a_i$ in the preceding formulas which temper the enthusiasm which was aroused in the minds of physicists when the situation described above was first discovered by Schroedinger.

6. *The Dirac Equations.* On one occasion before, in pursuing the analogy between analytic functions and wave mechanics, we reversed our steps; in analytic functions we used the substitution $w = Pe^{ik\theta}$ to eliminate a variable; in wave mechanics we used it to introduce a variable ($t$). We come now to another case of that nature. In analytic functions we arrive at the Laplace equation from a system of first-order equations. Here we already have a (generalized) Laplace equation and want to arrive from it at a system of first-order equations of which it is a consequence. Strictly speaking, we already have a solution of this problem. We saw that as a result of elimination of all but one component from Maxwell's system we arrive at a wave equation. It was also noted that the introduction of Maxwell's equations in optics may be considered as such a reversal from a second-order equation to a set of first-order equations. Dirac* achieved the same transition in wave mechanics, but he found another very remarkable set of first-order equations which lead to the wave equation. There are many attempts to substitute, for his equations, equations very closely related to Maxwell's equations, but it is not clear at present whether such an attempt will ever prove successful. We shall now explain Dirac's method. We strip the equation (46) of its unessential features, that is, the potentials and the mass-term. What remains we write in the form

$$(X^2 + Y^2 + Z^2 - T^2)w = 0, \tag{49}$$

where

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}, \quad T = \frac{\partial}{\partial t},$$

and try to find a first-degree operator whose square is

---

* P. A. M. Dirac, *Quantum theory of the electron*, first paper, Proceedings of the Royal Society, (A), vol. 117 (1928), pp. 610–624; second paper, ibid., vol. 118 (1928), pp. 351–361.

$X^2+Y^2+Z^2-T^2$. We write it in the form $\alpha X+\beta Y+\gamma Z+\delta T$, and expressing our requirement, we get

$$(50)\qquad \alpha^2=\beta^2=\gamma^2=-\delta^2=1,\ \alpha\beta+\beta\alpha=0,\ \text{etc.}$$

In order to simplify our discussion we shall consider the two-dimensional case. The solution of the equations $\alpha^2=\beta^2=1$, $\alpha\beta+\beta\alpha=0$ in numbers is impossible, of course, but we can find matrices, viz.,

$$(51)\qquad \alpha=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

which satisfy these conditions. *Formally*, then, the equation $(\alpha X+\beta Y)w=0$ is equivalent to $(X^2+Y^2)w=0$, that is, to the equation of Laplace; or, at the least, the solutions of the first equation must satisfy the second. But how shall we interpret the operator with matrices? Among many roles in which the elements of a matrix appear, one of the most important is the role of coefficients of a linear transformation. Since we have two-rowed matrices, we consider two-component vectors $(u, v)$. Now $\alpha$ transforms $(u, v)$ into $(u, -v)$; $\beta$ transforms $(u, v)$ into $(v, u)$. Our equation $(X\alpha+Y\beta)w=0$ gives then exactly the Cauchy-Riemann equations (1).

Returning to the four-dimensional case we would expect to arrive there at four-rowed matrices; this suggests four-component vectors, and that is what Dirac* and Darwin† found. After the equations have been obtained, we put back the potentials and the mass constant $A$ where they belong. It is important to note that, without potentials, the transition from one second-order equation to four first-order equations would be only a change of form; it would not affect the physical results. It *does* make a great deal of difference, however, whether we put in the potentials before or after the transition. Dirac's theory fits the experimental facts still better than Schroedinger's. The transition from Schroedinger's theory to that of Dirac plays the same role as the introduction of spin in the Bohr theory. It was noted

---

* Dirac, loc. cit.

† C. G. Darwin, *The wave theory of the electron*, Proceedings of the Royal Society, (A), vol. 118 (1928), pp. 654–680; *On the magnetic moment of the electron*, ibid., vol. 120 (1928), pp. 621–631.

that the potentials play a role analogous to that played by boundary conditions in other cases, and it has been mentioned in connection with the introduction of Maxwell's equations in optics that the transition from second-order to first-order equations there was also important in connection with its effect on boundary conditions.

We now write out Dirac's equations essentially in the form given them by Darwin:

$$
(52)\qquad \begin{cases} (p_4 + A)\psi_1 + (p_1 - p_2)\psi_4 + p_3\psi_3 = 0, \\ (p_4 + A)\psi_2 + (p_1 + ip_2)\psi_3 - p_3\psi_4 = 0, \\ (p_4 - A)\psi_3 + (p_1 - ip_2)\psi_2 + p_3\psi_1 = 0, \\ (p_4 - A)\psi_4 + (p_1 + ip_2)\psi_1 - p_3\psi_2 = 0, \end{cases}
$$

where

$$
p_4 = i\frac{\partial}{\partial t} + i\phi_4,\ p_1 = \frac{\partial}{\partial x} + \phi_1,\ p_2 = \frac{\partial}{\partial y} + \phi_2,\ p_3 = \frac{\partial}{\partial z} + \phi_3.
$$

In these equations we split up the complex quantities into real and imaginary parts, using the notations

$$
(53)\quad \psi_1 = N + iP,\ \psi_2 = L + iM,\ \psi_3 = iZ - U,\ \psi_4 = iY - X.
$$

Then the equations become

$$
(54a)\qquad \begin{cases} \dfrac{\partial U}{\partial x} + \dfrac{\partial Z}{\partial y} - \dfrac{\partial Y}{\partial z} - \dfrac{\partial L}{\partial t} \\ \qquad + AM + Z\phi_1 - U\phi_2 - X\phi_3 + M\phi_4 = 0, \\ -\dfrac{\partial Z}{\partial x} + \dfrac{\partial U}{\partial y} + \dfrac{\partial X}{\partial z} - \dfrac{\partial M}{\partial t} \\ \qquad - AL + U\phi_1 + Z\phi_2 - Y\phi_3 - L\phi_4 = 0, \\ \dfrac{\partial Y}{\partial x} - \dfrac{\partial X}{\partial y} + \dfrac{\partial U}{\partial z} - \dfrac{\partial N}{\partial t} \\ \qquad - AP + X\phi_1 + Y\phi_2 + Z\phi_3 + P\phi_4 = 0, \\ \dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z} + \dfrac{\partial P}{\partial t} \\ \qquad + AN - Y\phi_1 + X\phi_2 - U\phi_3 + N\phi_4 = 0, \end{cases}
$$

$$(54b)\quad \left\{\begin{aligned}
&\frac{\partial P}{\partial x}+\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}+\frac{\partial X}{\partial t}\\
&\qquad +AY+N\phi_1-P\phi_2-L\phi_3-Y\phi_4=0,\\
&-\frac{\partial N}{\partial x}+\frac{\partial P}{\partial y}+\frac{\partial L}{\partial z}+\frac{\partial Y}{\partial t}\\
&\qquad -AX+P\phi_1+N\phi_2-M\phi_3+X\phi_4=0\\
&\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}+\frac{\partial P}{\partial z}+\frac{\partial Z}{\partial t}\\
&\qquad +AU+L\phi_1+M\phi_2+N\phi_3-U\phi_4=0,\\
&\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}+\frac{\partial U}{\partial t}\\
&\qquad +AZ-M\phi_1+L\phi_2-P\phi_3-Z\phi_4=0.
\end{aligned}\right.$$

The first thing we notice about these equations is that they are a generalization of Maxwell's equations (25). In fact, if we make $U=P=0$, and also make all the potentials and the mass constant zero, we get exactly Maxwell's equations. Let us now keep $U$ and $P$ general but neglect the potentials and the mass constant, that is, neglect the last five terms in each equation. The resulting system is also a generalization of Maxwell's system, and therefore of the Cauchy-Riemann system (1). It has been investigated from this point of view by Iwanenko and Nikolski,* who consider all the eight components of Dirac as the components of a biquaternion, and, extending the investigation of Hanni,† obtain the equations as *integrability conditions*, that is, write certain integrals whose vanishing is equivalent to the original equations. It may be noted that these equations do *not* come under the Volterra theory.

A very interesting point about these equations is that although they are relativistically invariant, (that is, if we pass to another coordinate system, we may express exactly the same thing the equations tell us by a system of equations of the same type, with different values of the components, of course) the re-

* D. Iwanenko and K. Nikolski, Zeitschrift für Physik, vol. 63 (1930), p. 129.

† Loc. cit.

lations between the values of components in one system of coordinates and the values in another system, that is, the transformation formulas, are *not* of the type considered in ordinary tensor analysis. Much has been made of this circumstance. Weyl* connected it in a very beautiful way with the theory of representation of groups. Van der Waerden† invented a new tensor analysis, called spinor analysis, that takes care of the situation in a satisfactory way.

The situation takes a different aspect, however, when we introduce the *second-order quantities*. In the Schroedinger theory there was only one complex-valued component. We considered only one real-valued quadratic expression in it, namely its norm (see (48)). Now, as the result of the transition to first-order equations, we have many more components. Quite early in the development of the subject, in the first papers by Dirac and Darwin, a four-vector, two scalars, and a six-vector were introduced, whose components are quadratic in the $\psi$'s, and are transformed according to the ordinary formulas of tensor calculus. Further investigation showed the existence of still another four-vector whose components are quadratic in the $\psi$'s. As a result of the fact that all these components are expressed in terms of the four $\psi$'s there exist between them many relations. Limiting our attention to the two four-vectors, we find that they are perpendicular and of equal length.‡ This reminds us of analytic functions; we ask ourselves whether the divergences of these vectors vanish and find that they *do*, in the case when the mass constant is zero. We have thus in the Dirac theory two equal and perpendicular vectors with vanishing divergences. The theory seems then to provide a method of studying such situations, which, as we saw before (beginning of §2), involve non-linear relations, by means of *linear* equations, because the Dirac equations are linear. The method is based on introducing auxiliary quantities in such a way that the quadratic relations are the result of the fact that the two vectors are expressed in terms of the same auxiliary quantities. Taking this hint we find

* H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, 1928, §§25 and 39.

† B. L. van der Waerden, Göttinger Nachrichten, 1929, p. 100.

‡ O. Laporte and G. E. Uhlenbeck, Physical Review, vol. 37 (1931), pp. 1396 and 1553.

without difficulty that the components of two equal and perpendicular vectors in ordinary three-space (14) may be expressed as follows:

$$(55)\quad \begin{cases} f_1 = YZ - XU, & r_1 = \frac{1}{2}(X^2 + Z^2 - Y^2 - U^2), \\ f_2 = \frac{1}{2}(X^2 + Y^2 - Z^2 - U^2), & r_2 = XU + YZ, \\ f_3 = XZ + YU, & r_3 = ZU - XY. \end{cases}$$

If we subject these vectors to the condition (16) that their divergences should vanish, we obtain for the auxiliary quantities, $X$, $Y$, etc., a system of equations, which, although not linear, is a consequence of the linear system:

$$(56)\quad \begin{array}{ll} \dfrac{\partial X}{\partial x} + \dfrac{\partial U}{\partial y} - \dfrac{\partial Y}{\partial z} = 0, & \dfrac{\partial Y}{\partial x} - \dfrac{\partial Z}{\partial y} + \dfrac{\partial X}{\partial z} = 0, \\ \dfrac{\partial Z}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial U}{\partial z} = 0, & \dfrac{\partial U}{\partial x} - \dfrac{\partial X}{\partial y} - \dfrac{\partial Z}{\partial z} = 0, \end{array}$$

which in many ways is similar to the Dirac system with potentials and mass constant neglected.

In conclusion, a word concerning the potentials. As in the other cases in which we considered second-degree quantities, we may ask the questions whether the first-degree quantities are determined by the second-degree quantities, and what differential conditions are imposed on the second-degree quantities by the differential equations to which the first-order quantities are subjected. The answer to the first question is that the first-degree quantities are not completely determined by the second-degree quantities, but are determined to within a variable angle, just as $(u, v)$ are not determined by $u^2+v^2$, or $r$ and $f$ by $E$. In answering the second question, we have to eliminate therefore this variable angle, just as in the cases we had before. Without entering into details we may state that if we start with Dirac's equations without potentials we obtain in a way analogous to that by which we obtained (10) and (41), a system of exactly the form of the Dirac equations with potentials, except that the $\phi$'s are the derivatives of the arbitrary angle.

THE UNIVERSITY OF MICHIGAN