NOTE ON A THEOREM OF BÔCHER AND KOEBE*

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1. *Introduction.* In this paper a generalization of the following theorem, discovered independently by Bôcher† and Koebe,‡ is established.

THEOREM 1. *If $u(x, y)$ is continuous with its first partial derivatives in a plane region R, and if, for every circle C contained in R,*

$$\int_C \frac{\partial u}{\partial n} ds = 0,$$

where n is the exterior normal to C, then u is harmonic in R.

The generalization obtained is embodied in Theorem 2.

THEOREM 2. *If $v(x, y)$ is harmonic and positive in R, if $u(x, y)$ is continuous with its first partial derivatives in R, and if*

$$\int_C v \frac{\partial u}{\partial n} ds = \int_C u \frac{\partial v}{\partial n} ds \tag{1}$$

for every circle C contained in R, then u is harmonic in R.

Taking v as the constant one in Theorem 2, Theorem 1 is obtained.

Like Theorem 1,§ Theorem 2 has an analog in space, but,

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† Bôcher, M., *On harmonic functions in two dimensions*, Proceedings of the American Academy of Arts and Sciences, vol. 41 (1906), pp. 577–583.

‡ Koebe, P., *Herleitung der partiellen Differentialgleichung der Potentialfunktion aus der Intergraleigenschaft*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, vol. 5 (1906), pp. 39–42.

§ Koebe, loc. cit. For generalizations of Bôcher's and Koebe's Theorem of another type, see G. C. Evans, *Fundamental points of potential theory*, Rice Institute Pamphlets, vol. 7 (1920), pp. 252–329, espe ially p. 286, and *Note on a theorem of Bôcher*, American Journal of Mathematics, vol. 50 (1928), pp. 123–126; and G. E. Raynor, *On the integro-differential equation of the Bôcher type in three space*, this Bulletin, vol. 52 (1926), pp. 654–658. Evans, using the notion of the potential function of a gradient vector, shows that the conclusion of Theorem 1 holds with much lighter hypotheses both on u and the character of the curves C.

since no new essentially different details present themselves in the proof for space, we simply state this analog, and consider in detail only the plane case.

THEOREM 3. *If $v(x, y, z)$ is harmonic and positive in a region R in space, if $u(x, y, z)$ is continuous with its first partial derivatives in R, and if, for every sphere C contained in R,*

$$\int\int_C v\frac{\partial u}{\partial n}ds = \int\int_C u\frac{\partial v}{\partial n}ds,$$

where n is the exterior normal to C, then u is harmonic in R.

The proof of Theorem 2 is elementary in character. The idea is to express uv as a sum of integrals and deduce the character of u from the properties of these integrals.

2. *Proof of Theorem* 2. We first observe that it is enough to prove the theorem in the case that R is the interior of a circle C, and the hypotheses hold in the interior R' and on the boundary* of a circle C' concentric with C but of larger radius. The problem, then, is to show that u_{xx} and u_{yy} exist and are continuous in R, and that

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{2}$$

there.

Let $P(x, y)$ be any point in R. Let α' be the radius of C', α the radius of C, and

$$\rho = \tfrac{1}{2}(\alpha' - \alpha).$$

Then, by (1), the hypothesis on v, and a classical formula, we have, for $0 < t \leqq \rho$,

$$\int_{C(P,t)} v\frac{\partial u}{\partial n}ds = \int_{C(P,t)} u\frac{\partial v}{\partial n}ds = \int\int_{\sigma(P,t)} \phi d\sigma, \tag{3}$$

where $\sigma(P, t)$ is the interior of the circle $C(P, t)$ of radius t about P, and

$$\phi(x, y) = \nabla u\cdot\nabla v = u_x v_x + u_y v_y.$$

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* That is to say, the hypotheses hold in a region containing R' and its boundary.

It is the third integral in (3) that enables us to express uv as a sum of integrals, whose properties lead to the conclusion of the theorem. Writing

$$\sigma = \sigma(P, \rho), \quad S' = R' + C', \quad r = \left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2},$$

we find that

$$\begin{aligned}
&\pi\rho^2 u(x, y)v(x, y)\\
&\quad = \left\{ \iint_\sigma uv\,d\sigma + \rho^2\left(\frac{1}{2} + \log\frac{1}{\rho}\right)\iint_\sigma \phi\,d\sigma \right.\\
&\qquad \left. - \frac{1}{2}\iint_\sigma \phi(\xi, \eta)r^2 d\xi d\eta - \rho^2 \iint_{S'-\sigma} \phi \log r\,d\sigma \right\} \qquad (4)\\
&\qquad + \rho^2 \iint_{S'} \phi \log r\,d\sigma\\
&\quad = J'(P) + J''(P), \text{ say.}
\end{aligned}$$

In fact, by (3),

$$\begin{aligned}
\int_0^\rho \tau d\tau \int_0^\tau \frac{dt}{t}\int_{C(P,t)} v\frac{\partial u}{\partial n}\,ds &= \int_0^\rho \tau d\tau \int_0^\tau \frac{dt}{t}\int_{C(P,t)} u\frac{\partial v}{\partial n}\,ds\\
&= \int_0^\rho \tau d\tau \int_0^\tau \frac{dt}{t}\iint_{\sigma(P,t)} \phi\,d\sigma,
\end{aligned}$$

or $K_1(P) = K_2(P) = K_3(P)$, say. Introducing, then, a system of polar coordinates (r, θ) with pole at P, we have

$$\begin{aligned}
K_1(P) &= \int_0^\rho \tau d\tau \int_0^{2\pi} d\theta \int_0^\tau v\frac{\partial u}{\partial r}\,dr\\
&= \int_0^\rho \tau d\tau \int_0^{2\pi}\left\{uv - u(x, y)v(x, y) - \int_0^\tau u\frac{\partial v}{\partial r}dr\right\}d\theta\\
&= \iint_\sigma uv\,d\sigma - \pi\rho^2 u(x, y)v(x, y) - K_2(P),
\end{aligned}$$

so that

$$\pi\rho^2 u(x, y)v(x, y) = \iint_\sigma uv\,d\sigma - 2K_3(P).$$

But, we see that

$$K_3(P) = \int_0^{2\pi} d\theta \int_0^{\rho} \tau d\tau \int_0^{\tau} \frac{dt}{t} \int_0^t r\phi dr$$

$$= \int_0^{2\pi} d\theta \int_0^{\rho} \left\{ \tfrac{1}{2}\rho^2(\log \rho - \tfrac{1}{2}) + \tfrac{1}{4}r^2 - \tfrac{1}{2}\rho^2 \log r \right\} r\phi dr,$$

upon changing the order of integration twice. Hence (4) follows.

Consider, now, the derivatives of J'. We have

$$J'(P) = \int_{y-\rho}^{y+\rho} d\eta \int_{x-\psi}^{x+\psi} w d\xi + \left\{ \int_{-\alpha'}^{y-\rho} d\eta \int_{-\psi'}^{\psi'} + \int_{y-\rho}^{y+\rho} d\eta \int_{-\psi'}^{x-\psi} \right.$$

$$\left. + \int_{y-\rho}^{y+\rho} d\eta \int_{x+\psi}^{\psi'} + \int_{y+\rho}^{\alpha'} d\eta \int_{-\psi'}^{\psi'} \right\} w' d\xi,$$

where

$$w = w(x, y; \xi, \eta) = u(\xi, \eta)v(\xi, \eta) + \phi(\xi, \eta)\left\{\tfrac{1}{2}\rho^2 - \rho^2 \log \rho - \tfrac{1}{2}r^2\right\},$$

$$w' = w'(x, y; \xi, \eta) = -\rho^2\phi(\xi, \eta) \log r,$$

$$\psi = \psi(\eta, y) = \left\{\rho^2 - (\eta - y)^2\right\}^{1/2}, \psi' = \psi'(\eta) = (\alpha'^2 - \eta^2)^{1/2}.$$

We see, then, by using the fact that u, v and ϕ are continuous in S', and the formula for differentiation under the integral sign, that, for P in R, J_x' exists and is given by

$$J_x' = \int_{y-\rho}^{y+\rho} \left\{ w(x, y; x + \psi, \eta) - w(x, y; x - \psi, \eta) \right\} d\eta$$

$$+ \int_{y-\rho}^{y+\rho} \left\{ w'(x, y; x - \psi, \eta) - w'(x, y; x + \psi, \eta) \right\} d\eta$$

$$+ \iint_{\sigma} w_x d\sigma + \iint_{S'-\sigma} w_x' d\sigma$$

$$= \int_{y-\rho}^{y+\rho} \left\{ u(x + \psi, \eta)v(x + \psi, \eta) - u(x - \psi, \eta)v(x - \psi, \eta) \right\} d\eta$$

$$- \iint_{\sigma} (x - \xi)\phi d\sigma - \rho^2 \iint_{S'-\sigma} \phi \frac{(x - \xi)}{r^2} d\sigma;$$

which reduces to

$$(5)\quad J_x'=\iint_\sigma\{uv_x+vu_x-(x-\xi)\phi\}\,d\sigma-\rho^2\iint_{S'-\sigma}\phi\frac{(x-\xi)}{r^2}\,d\sigma,$$

upon writing

$$u(x+\psi,\eta)v(x+\psi,\eta)-u(x-\psi,\eta)v(x-\psi,\eta)$$
$$=\int_{x-\psi}^{x+\psi}\{uv_x+vu_x\}\,d\xi.$$

From (5) and the continuity of u and v and their first partial derivatives, we deduce immediately that J_{xx}' and, as it is worth while noting for future purposes, J_{xy}' exist and are continuous in R. By analogy, J_{yy}' exists and is continuous in R.

Next, consider J''. The existence of J_x'' and J_y'' can be inferred from (4) and the existence of J_x', J_y' and the first partial derivatives of u and v. We wish to know, further, that J_{xx}'' and J_{yy}'' exist and are continuous in R. To prove this, we first observe that u, v^{-1}, J_x', J_y', J_x'', J_y'' satisfy uniform Hölder conditions* in any closed domain S'' bounded by a circle C'' contained in R. The first four of these functions have this property because their first partial derivatives are continuous in R and R is convex† and contains S'', the last two because of a theorem of Dini.‡ We next observe that from this property of u, v^{-1}, $\cdots$, J_y'', it follows that ϕ satisfies a uniform Hölder conditionin S'', for

$$\phi=\{\nabla v\cdot\nabla J'+\nabla v\cdot\nabla J''-\pi\rho^2 u\nabla v\cdot\nabla v\}/(\pi\rho^2 v),$$

and thus ϕ is equal to a combination of sums and products of functions each of which satisfies a uniform Hölder condition in

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* A function $f(P)$, defined on a set E, satisfies a uniform Hölder condition on E if, P and Q being any two points of E,

$$|f(P)-f(Q)|<A\,|PQ|^\lambda,$$

where A and λ are independent of P and Q, and $\lambda>0$. Evidently, if $f_1(P)$ and $f_2(P)$ satisfy uniform Hölder conditions on E, f_1+f_2 and f_1f_2 have the same property.

† A region R is convex if each segment, whose end points lie in R, lies in R.

‡ U. Dini, *Sur la méthode des approximations successives pour les équations aux dérivées partielles du deuxième ordre*, Acta Mathematica, vol. 25 (1901), pp. 185–230. The function J'' is, of course, the potential function due to a distribution of continuous density $-\rho^2\phi$ over S'.

S''. The existence and continuity of J''_{xx} and J''_{yy} can now readily be deduced. We write

$$\begin{aligned} J'' &= \rho^2 \iint_{S'-S''} \phi \log r d\sigma + \rho^2 \iint_{S''} \phi \log r d\sigma \\ &= L' + L'', \text{ say.} \end{aligned}$$

Now L'_{xx} and L'_{yy} evidently exist and are continuous for P in the interior of S'', while L''_{xx} and L''_{yy} exist and are continuous in S'' by a theorem of Hölder.* Thus J''_{xx} and J''_{yy} exist and are continuous in the interior of S''. But C'' was arbitrary in R; and hence it follows that J''_{xx} and J''_{yy} exist and are continuous in R.

The proof is now almost complete. Since J'_{xx}, J'_{yy}, J''_{xx}, J''_{yy} exist and are continuous in R, and since v is positive and harmonic there, u_{xx} and u_{yy} exist and are continuous in R. It remains, then, only to prove that (2) holds. To prove this, we show that a contrary assumption leads to a contradiction. Suppose there is a point P in R at which $|\nabla^2 u| = 2\beta$ is different from zero. Then we can choose t so small that $\sigma(P, t)$ lies in R and $|\nabla^2 u| > \beta$ in $\sigma(P, t)$. Thus, since v exceeds a positive constant ϵ in $\sigma(P, t)$ and $\nabla^2 u$ is continuous there,

$$(6) \qquad \left| \iint_{\sigma(P,t)} v \nabla^2 u d\sigma \right| \geqq \epsilon\beta \text{ area } \sigma(P, t) > 0.$$

But

$$(7) \qquad \iint_{\sigma(P,t)} v \nabla^2 u d\sigma = \int_{C(P,t)} v \frac{\partial u}{\partial n} ds - \int_{C(P,t)} u \frac{\partial v}{\partial n} ds = 0,$$

by the continuity of $\nabla^2 u$, our hypotheses, and Green's formula. In (6) and (7) we reach the desired contradiction. The proof is now complete.

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* Otto Hölder, *Beiträge zur Potentialtheorie*, Inaugural Dissertation, Stuttgart, 1882, p. 17. Hölder considers the second partial derivatives of a volume distribution. The same type of analysis, however, holds for plane distributions.