THE PROBLEM OF THE CUBIC VARIETY IN S_4*

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1. *Introduction.* One of the outstanding problems of algebraic geometry that is still unsolved is the classification of Cremona transformations in space of more than two dimensions, and another, closely related to the preceding, is that of possible series of composition.

The theorem of Noether (80), later clarified and interpreted by Segre (90), Castelnuovo (14), Chisini (17), and Alexander (1), has answered the latter problem for the plane; while a large number of papers, represented in particular by the recent work of Hudson (67), Mlodzieioski (73)–(76), and Montesano (79), have made the construction of tables a definite procedure.

The smaller problem of the involutorial transformations has been completely solved by Noether (81), Bertini (6), and others, thus furnishing a weapon of incalculable importance for the study of various applications. Periodic transformations and their groups have been studied from various points of view by Kantor (69)–(70), Wiman (112), and Coble (20)–(21).

The corresponding problem in space is still almost undeveloped. Notwithstanding the excellent report made by Coble (22), the less ambitious one by me (103), the appearance of the extensive treatise by Hudson (67), and the Report of a committee of the National Research Council (106), I wish to speak of one phase of the latter problem. For regular transformations in space of three dimensions, the important theorem of Hudson (66) and a recent memoir of Montesano (78) are distinct steps in advance. When the

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defining elements are chosen independently, without contact conditions, the table of characteristics for a Cremona transformation and for its inverse has been completely constructed.

I wish to confine my attention, however, to the part taken by the cubic variety of four-way space S_4 in the study of involutorial birational transformations of S_3.

The study of the involution defined by the web of quadrics through six points leads at once to the Weddle surface, the locus of the vertex of the quadric cones through the six points, as surface of coincident points. If the quadrics of the web be made projective with the planes of another space S_3', a (1, 2) correspondence is established, and the Kummer surface appears as surface of branch points in S_3'. All the properties of the general transformation and of all particularizations can be made from this point of view. This study of (1, 2) correspondences is an extension of that initiated by De Paolis (27) for S_3, and applied to double planes by Enriques (32) and by Castelnuovo and Enriques (15), thus confirming the result obtained by Noether (81) concerning the number of plane involutions of order two.

But the Kummer surface is the apparent contour of the cubic variety with ten nodes, as has been shown by Segre and further developed by Hudson (68) and by Snyder (101). Moreover, the same surface also appears in a similar rôle in connection with involutions defined by webs of cubic and of certain quartic surfaces. See Sharpe and Snyder (100).

In the same way, the quartic surface with less than sixteen nodes, birationally equivalent to the focal surface of line congruences of order two and class 3, 4, 5, 6 appear as surfaces of branch points of certain involutions, and each is the apparent contour of a cubic variety with one or more actual double points. Thus, to a point on the cubic variety correspond a pair of points in S_3, each point of which uniquely determines the other. These pairs of points determine an involution belonging to V_3.

2. *Rational Involutions.* Given an algebraic curve C in S_n, a necessary and sufficient condition that C is rational is that its points can be put into (1, 1) continuous correspondence with those of a straight line. (Definition.)

If $\rho x_i = f_i(y_1, y_2)$, $(i=0, 1, \cdots, n)$, define the coordinates (x) of a point on $C(x)$ and y_1, y_2 those of the image point on the line, then f_i is a rational integral polynomial homogeneous in y_1, y_2.

If several sets of values of y_1, y_2 define the same variable point (x), then functions $y_1 = \phi_1(y_1', y_2')$, $y_2 = \phi_2(y_1', y_2')$ can be found such that the relation between (y') and (x) is (1, 1). This is expressed by saying that *every involution on a rational curve is rational.* (Theorem of Lüroth (71)).

Similarly, given an algebraic surface $F(x)$ in S_n, such that $\rho x_i = f_i(y_1, y_2, y_3)$, wherein each f_i is a rational integral form of the same degree in (x), then $F(x)$ is rational. If several sets of values of (y) define the same variable (x), then functions $y_i = \phi_i(y_1', y_2', y_3')$ can be found such that the relation between (y') and (x) is (1, 1). Hence *every involution on a rational surface or in particular on a plane, is rational.* (Theorem of Castelnuovo (11).) The theorem for involutions of order two was already known. See Noether (81) and Bertini (6). More recent contributions to the classification of plane involutions of order higher than two have been made by Howe (65), Hollcroft (64), and Sharpe (99).

Given a three-dimensional manifold $M_3(x)$ and $\rho x_i = f_i(y_1, y_2, y_3, y_4)$ a parametric representation, wherein f_i is a rational integral form, we cannot conclude whether M_3 is rational or not. The condition is necessary but it is not sufficient. If various sets of values (y) define the same variable set (x), it may not be possible to find a set of functions $y_i = \phi_i(y_1', y_2', y_3', y_4')$ such that the relation between (x) on $M_3(x)$ and (y') in S_3' is (1, 1). (Theorem of Enriques (36).)

Illustrations have been given of irrational involutions in S_3 belonging to a three-dimensional variety; others were furnished by Aprile (2) but all are of order larger than two. No illustrations have been found of three-dimensional involutions of order two that are known to be irrational. In particular, it is not known whether the points of the general cubic variety V_3 in S_4 can be put into (1, 1) correspondence with those of S_3.

It is known that an involution of order two in S_3 belongs to it. The existence of this involution was known to Noether, and later proved by Segre (89) and by Marletta (72). The actual form of the parametric represention is

$$\begin{aligned}
\rho x_1 &= y_1^2(2y_1^3 y_3 y_4 + A y_3 + B y_1),\\
\rho x_2 &= y_1 y_2(\quad \text{“} \quad),\\
\rho x_3 &= - y_3^2(\quad \text{“} \quad),\\
\rho x_4 &= y_1(y_1^3 y_3 y_4^2 + B y_1 y_4 - C y_3),\\
\rho x_5 &= - y_1(y_1\, y_4^2 + A y_1 y_4 + C y_1),\\
V_3 &= x_1 x_4^2 + x_3 x_5^2 + a x_4 + b x_5 + C = 0,
\end{aligned}$$

$$\begin{aligned}
a &= a_0 x_3^2 + a_1 x_3 + a_2, & A &= a_0 y_3^4 - a_1 y_3^2 y_1 + a_2 y_1^2,\\
b &= b_0 x_3^2 + b_1 x_3 + b_2, & B &= b_0 y_3^4 - b_1 y_3^2 y_1 + b_2 y_1^2,\\
c &= c_1 x_3^2 + c_2 x_3 + c_3, & C &= c_1 y_3^4 - c_2 y_3^2 y_1 + c_3 y_1^2,
\end{aligned}$$

where a_i, b_i, c_i are binary forms of order i in x_1 and x_2.

This form of order 7 is the simplest one consistent with a quadratic locus of invariant points. This result was obtained and the associated involution of order 6 is described by Snyder (104).

3. *Involutions Belonging Multiply to a Line Complex.* Every V_3 contains ∞^2 straight lines. See Enriques (28) and Fano (57)–(58). Let l be a line on V_3, and P any point on V_3.

The plane (P, l) meets V_3 in a residual conic through P which meets l in two points Z_1, Z_2. The line PZ_i is tangent to V_3 at Z_i. Conversely, any line tangent to V_3 at a point Z on l meets V_3 again in a point P.

Between the points P of V_3 and the tangents at points on l there is therefore a (1, 2) correspondence. These tangent lines form a special linear line complex which can be mapped birationally on the points of an S_3. The lines joining a pair of associated points PP' in S_3 always meet the rectilinear image of l; hence they define a special linear complex. But every line of the complex contains two pairs of associated points. (Snyder (104).)

Other I_2 in S_3 exist which are contained multiply in a rational complex. The simplest one is obtained by constructing a (1, k) correspondence between a pair of lines on a regulus and the quadrics of a pencil. A point P of S_3 determines a quadric passing through it, and hence a pair of directrices (d, d') of the regulus. The transversal of (d, d') through P meets the quadric in P'. The line PP' contains k pairs of associated points, and belongs to a general linear complex. (Snyder (107).)

Consider a line l and a pencil of surfaces of order n having the line to multiplicity $n-2$. Make the same (1, k) correspondence between the points of the line and the surfaces of the pencil. The line may be replaced by a conic provided the surface is of order less than 5, or by a space cubic.

A pencil of quadrics and a rational curve of any order may be taken. Similarly, a linear system consisting of a line and of a rational curve of order m meeting it in $m-1$ points may be put in (1, k) correspondence with the quadrics of a pencil. Either the line or the curve may remain fixed. The system may be replaced by a pencil of cubic curves as directrices. All of these forms contain types which belong to the cubic variety as particular cases. Incidentally they all possess properties not shared by other involutions heretofore known.

Another illustration is that of order 7 found by Montesano (77). Given five linear line complexes K_i and a projective form q or g of five elements. An arbitrary plane π of space has five poles O_i as to K_i, and in it is one point O' such that the lines $O'O_i \,\overline{\wedge}\, g$. There is one and only one complex K in involution with every K_i. The plane π has a pole O as to K. The relation between O, O' is birational and involutorial.

The line OO' describes a linear complex K and every line of K contains three pairs of conjugate points. There is no surface of invariant points, but a curve of order 8 and genus 5, birationally equivalent to the only fundamental curve of the system, apart from parasitic lines. It is not known whether this involution belongs to the cubic variety or not. Its properties are different from those of the other types just mentioned.

In particular, if the set of complexes be considered, such that each new one is in involution with five of the given ones (K_i, K), and the corresponding transformation be considered, the rôles of the fundamental curve and of the curve of invariant points will be interchanged. The scheme really furnishes ∞^2 pairs of such complementary involutions, corresponding to the ∞^2 possible values of g.

4. *Infinite Discontinuous Groups.* Various infinite discontinuous groups which belong to V_3 exist. We shall consider six distinct generating operations numbered (a), (b), (c), (d), (e), (f) below. It has not been shown whether others also exist.

(a). *The projection of V_3 upon itself from a vertex A upon it.* The product AB of two such projections is not periodic, nor can it be replaced by another product. Similarly for the product of three or more such projections.

The projection A is Cremonian, and quadratic. An S_3 is transformed into a quadric variety having the polar quadric of A for locus of invariant points. The hyperplanar sections of V_3 are transformed into sextic surfaces all passing through the six lines of V_3 which pass through A. These lines have the property that the image of a point on any one of them is the whole line passing through it.

In the product AB the S_3 sections of V_3 are transformed into surfaces of order 12, having six double basis lines through B, and six conics, all lying on a quartic surface, through another point, the image of A in the operation B. Similarly for products of more projections $A, B, C \cdots$. (Snyder (102).)

(b). *Let l be a fixed line on V_3. A point P of V_3 and l determine a plane which meets V_3 in a residual conic passing through P.*

The line l has a pole L as to this conic. The harmonic homology defined by L and l leaves the conic, and hence also V_3, invariant. The transformation is Cremonian, of order 8. The conjugate of an S_3 contains l to multiplicity 7. The locus of invariant points is a variety of order 5, containing l to multiplicity 4. The image of l itself is a variety of order 7, containing l to multiplicity 6. Every S_3 through l is transformed into itself.

There are ∞^2 lines on V_3, each of which has an associated involution (b). The product of two such involutions, associated with different lines, is not periodic, nor can one such product be replaced by another when the lines are taken arbitrarily.

The product of a transformation of the type (a) and a transformation of the type (b) is in general not periodic. (Snyder (102).)

(c). *Consider a pencil of* S_3 *having* π: $(ax)=0$, $(bx)=0$ *for base, and the line* $l\equiv x_1=x_2=x_3=0$ *on* V_3. Make the points of the line and the hyperplanes of the pencil projective, by associating each S_3 with the point P in which it meets l. Project the cubic surface S_3, V into itself from P. The transformation is Cremonian, of order 4, the fundamental elements are l taken simply, and the basis plane taken twice. The locus of invariant points is a cubic variety containing l, π each simply.

A transformation of this kind exists for every line l on V, and when l has been fixed, the basis plane may be chosen in ∞^6 ways. The product of two or more transformations of this type is not periodic. The product of any transformation of type (c) with one of type (a) or of type (b) or any sequence of (a), (b), (c) is in general not periodic.

The line l may be replaced by a conic, provided the basis plane has one point on it. The variety V_3 contains ∞^4 conics. Moreover, a space cubic with two points on the basis plane or a rational quartic with all three basis points upon it, may be taken. The directrix curve appears as a basis curve with multiplicity equal to its order in every case.

(d). *Assume an arbitrary fixed line m through a point M on V_3, but not lying on V_3.* A point P of V_3 and the line m determine a plane which meets V in a cubic curve through M. The first tangential M' of M, joined to P, meets V_3 in P'. The transformation P, P' is involutorial and Cremonian of order 8. The locus of M' is the surface of intersection of V_3 and the three-dimensional tangent space at M, of order 3. Every plane through m is transformed into itself. The transformation in each such plane is quadratic, a perspective Jonquières type of the third kind. The fundamental line is the tangent to the cubic curve at M', and the locus of invariant points is the polar conic of M' as to the cubic. The images of the hyperplane sections $|x|$ of V_3 are surfaces of order 24, $|8x|$, having M to multiplicity 6, and the six lines of V through M to multiplicity 2. They also contain the locus of the first tangential of M' as the plane describes the linear system through m.

The product of two transformations of type (d) is not periodic. There are ∞^3 points on V_3, and ∞^3 directions m associated with each point. The properties of this transformation should be further studied, both in S_4 and in S_3. Many of them are different from those of involutions heretofore known. The product of a transformation of type (d) with any one of a preceding type or one derived from any combination of the preceding types is in general not periodic.

(e). *Let l, l' be two skew lines on V_3 and π a plane not meeting either.* The plane is the basis of a pencil of S_3. The S_3 through a point P meets (l, l') in (K, K'). The line (K, K') meets V in M. The line PM meets V_3 in P'. The transformation is involutorial, and consists in interchanging P and P'. The locus of M is a rational C_4. The lines l and l' lie in a space Σ_3 which cuts V_3 in a cubic surface F_3 containing l and l'. The pencil (S_3) meets (l, l') in ranges projective with it, hence with each other, hence (K, K') describes a quadric surface H_2 in Σ_3, through (l, l'). The residual intersection of F_3 and H_2 is a rational C_4 which meets (l, l') each in three points.

Either or both lines may be replaced by conics skew to each other, provided π meets each conic in one point, or one may be replaced by a cubic with two points on π, with the other directrix either a line skew to π or a conic with one point on π. When both directrices are cubics, they must have a common point on π. Corresponding forms exist in a three-way space which leave a given cubic surface invariant, each of which presents interesting configurations of fundamental elements.

(f). Another transformation is the following. *Given a line l on V_3 and a plane π of S_4, skew to l.*

The points O of π and the planes through l are projective, and perspective. The plane through l meets V_3 in a residual conic. Consider the harmonic homology having O for center and its pole as to the conic in its associated plane for axis. This procedure defines an involutorial birational transformation which leaves each conic of l in a plane through l invariant, and hence transforms V_3 into itself.

It is Cremonian. The line l and the plane cubic curve section of V_3 by π are the fundamental elements. The plane π may be replaced by any rational surface meeting an arbitrary plane through l in one point apart from those on l.

5. *Continuous Groups.* A curve of genus zero belongs to a three-parameter group of continuous transformations. One of genus 1 has ∞^1 transformations which do not form a group. If the genus exceeds unity, the curve belongs at most to a finite group. Surfaces invariant under ∞^1 linear transformations are rational or reducible to ruled surfaces. If they are invariant under ∞^2 such transformations they are necessarily rational. (Fano (40).) Every algebraic surface, invariant under a transitive continuous group of linear transformations, can be transformed birationally into a plane, or a quadric of S_3, or into a rational cone normal in a certain S_{m+1} in such a manner that the group considered gives rise respectively to a group of plane homographies, or to a group of homographies in S_3, or in S_{m+1}, each leaving the associated form invariant. (Fano (41) and (44).)

Every three-parameter linear group of S_3 leaves a series of forms generated by planes invariant, none containing a simpler one. (Fano (40).)

The three-dimensional varieties of S_4 invariant under ∞^4 continuous linear transformations are classified into five types, which do not include the general cubic variety. (Fano (43).)

Surfaces invariant under a continuous group of Cremona transformations are all reducible to rational surfaces in S_3 invariant under linear transformations. (Fano (45).) Similarly for three-dimensional varieties. (Fano (46).)

Given a homogeneous linear differential equation of order n. If a system of solutions $y_1, \cdots, y_n$ be regarded as projective coordinates of a point of S_{n-1}, the point will describe a rational curve when the coefficients in the given equation are rational functions of x. If the (y) satisfy one or more algebraic relations, the curve will lie on a corresponding number of varieties. The group of rationality of the equation leaves each variety of the system invariant. Finite groups of Cremona transformations in the plane, both according to the order and to the type, in the sense of Kantor and of Wiman have all been determined in this way. (Enriques and Fano (38).) The types of continuous groups in the plane are projective, quadratic and Jonquières, and in S_3 contain these and two other ∞^3 types. The generalized Jonquières groups leave a pencil of planes or a bundle of lines (or both) invariant. (47).

The imprimitive groups of continuous Cremona transformations in S_3 belong as subcases in the types derived by Enriques (29) and Fano (46), (48).

The complete enumeration of continuous Cremona transformations in S_3 can be made by starting from the generating infinitesimal transformation. This has been done and the preceding results confirmed. (49).

A differential equation of the type in question is always solvable when more than one algebraic equation is identically satisfied by the solutions. (50).

The group of rationality may leave a quadratic form in S_{n-1} invariant. (52). For $n=6$, this case can be interpreted in terms of lines of S_3; instead of a curve, the locus is now a ruled surface. For $n=5$, the same interpretation can be made by assuming a linear relation among six solutions. The ruled surface is then contained in a linear line complex. (51). If the discriminant of the quadratic form vanishes (special complex), the equation can be reduced to one of order 4. (54).

Every M_3 invariant under ∞^3 projective transformations is rational. If the group is not integrable, the existence of ∞^1 rational surfaces on it insures its rationality. (53).

These results have been collected and systematically developed by Fano (55). It is shown that the existence of algebraic relations among the fundamental solutions of a linear differential equation is a necessary and sufficient condition for the vanishing of certain invariants belonging to the equation.

On the general quartic three-dimensional variety V_3^4 of S_4, without double points, the system of hyperplane sections is the only linear system of regular surfaces with genera equal to unity and of dimension not less than 2. This variety is not invariant under any birational transformation, with the possible exception of linear ones.

The M_3^6 of S_5, complete intersection of V_4^2 and V_4^3, does not contain such surfaces either. Every birational transformation which leaves it invariant is the product of a finite number of double projections from lines lying on it and of possible linear transformations. (Fano (61).)

6. *Invariants*. The question naturally arises whether algebraic or topological invariants exist, particular values of which can indicate the rationality or irrationality of V_3.

A necessary and sufficient condition that an irreducible algebraic curve in space of any number of dimensions shall be rational is that its genus is zero. (Clebsch (18).)

An algebraic surface has an infinite number of invariants analogous to the genus of a curve, but a necessary and sufficient condition that a surface shall be rational is that its arithmetic genus p_n and its bigenus P_2 should both be zero. (Castelnuovo (13).) That a surface may have both its geometric genus and its arithmetic genus equal to zero and still not be rational, was shown by Enriques (35).

A system of invariants of three-dimensional varieties was determined by Pannelli (84), analogous to the genera of a surface. (Castelnuovo and Enriques (16).) Some of the fundamental concepts of invariants on hyperdimensional manifolds had been discussed by Noether (82). It was then shown by Severi (97) that all these invariants vanish for the cubic variety of S_4, also for the general quartic V_3^4 of S_4, and for the $M_3^6 \equiv (V_4^2, V_4^3)$ of S_5, hence that the various indices of irregularity are zero. These latter V_3^4, M_3^6 contain only complete intersections. (Severi (92), Fano (59).)

The algebraic surfaces contained in V_3^3 were obtained by Fano (56).

The M_3^6 is the representative of the general cubic line complex of S_3, which has been extensively studied by Voss (111) and by Veneroni (109). In particular, it has ∞^1 linear pencils of lines or quadric reguli, hence M_3^6 contains ∞^1 straight lines. By projecting the manifold from one of them, it can be mapped on S_3 doubly, with a sextic surface of branch points. The two forms, V_3^4 and M_3^6, are birationally distinct. (Fano (59).)

The assumption that either contains a homaloidal system of surfaces leads to a contradiction; hence both of these forms are irrational, that is, they can not be mapped birationally on S_3. (Fano (59).) However, both can be mapped on S_3 by means of an involution that is rational in one sense only.

The actual process was outlined for M_3^6 by Enriques (36), which establishes the important result that irrational involutions in S_3 exist. Other examples of involutions belonging to three-dimensional varieties were given by Marletta (72), and Aprile (2) showed that the Enriques involution is of order 216, but that the manifold can be mapped on S_3 by means of an involution of order 36. Further conditions for rationality were found by Enriques (37).

These forms are much more general than V_3^3. The general M_3^6 does not contain any planes, since the general cubic line complex of S_3 does not contain a bundle of lines. If now the complex be particularized to contain three distinct bundles of lines (or dually, three plane fields of lines), then the M_3^6 contains three planes belonging to the same system on the quadric variety passing through it. This particularized M_3^6 is birationally identical with the general V_3^3.

Similarly, the V_3^4 can be particularized to the extent of having a double line and still be much more general than the general cubic variety. It may be desirable to study the properties of these particular forms in order to obtain those of the general V_3^3.

The two general forms V_3^4 and M_3^6 are included in the general category M_3^{2p-2} of S_{p+1} (Fano (62)) having surface sections regular with all their genera equal to unity, and curve sections canonical curves of genus p. V_3^4 corresponds to $p=3$ and M_3^6 to $p=4$. The case $p=2$ appears as a double S_3 with a sextic surface of branch points, but this can be obtained by projecting M_3^6 upon S_3 from one of its lines, in a particular form.

The case $p=5$ is represented by the intersection of three general quadratic varieties in S_6. It contains no surfaces other than complete intersections with other forms. This manifold can be mapped on S_3 by an involution of order 4. It contains an infinite number of rational congruences of the first order, of rational curves.

The general $V_3^n : S_1^{n-2}$ of S_4 has been studied from various points of view, but most of the properties found for larger values of n do not exist for $n=3$. (Enriques (33).)

As yet no topological invariants have been found that are characteristic of the general V_3^3, normal in S_4.

Various properties regarding the analytical representation of a variety and of manifolds have been found by Severi

(91), those regarding postulation (93), those concerning complete intersections of non-singular forms (94), (95).

7. *Sections by* $|S_3|$. If all the plane sections of a surface in S_3 are rational, the surface is rational. It is either ruled or is a Steiner quartic surface with three concurrent double lines. (Picard (85), Guccia (63).)

If the sections by $|S_{n-1}|$ of a manifold M_3 in S_n are rational surfaces, M_3 is rational except possibly when M_3 is V_3^3 of S_4, concerning which no conclusion can be drawn, as the method does not apply. (Fano (60).)

Probably the transcendental methods of Bagnera-De Franchis (3)-(5), those of Enriques-Severi (3), and those of Severi (96), (98) as applied to hyperelliptic surfaces, may be extended to three-dimensional varieties. As yet no light has been thrown on V_3^3 from this source. Another possible method of extension is that of Comessatti (23) in his study of real rational surfaces.

8. *Rational* V_3^3. Let there be given a cubic variety $V_3^3(x) = 0$ in S_4, such that each x_i may be expressed in the form $\rho x_i = f_i(y_1, y_2, y_3, y_4)$, each f_i being a rational quaternary form of order N. When these values of x_i are substituted in $V(x) = 0$, it shall be identically satisfied, and that for an arbitrary set of values of the y_i; no other set can be found which will define the same value of (x).

Since $\sum a_i x_i = 0$, $V_3^3(x) = 0$ define a cubic surface, it follows that the $f_i(y)$ must satisfy the following properties.

(a) The system is linear of dimensionality 4.

(b) Any two surfaces of the system intersect in a curve of genus 1. (Plane sections of a cubic surface.)

(c) Any three intersect in three variable points.

From any simple point O_x on $V_3^3(x)$ project the variety on any space $S_3 \equiv \pi$ not passing through O_x. In π, every point is the image of two points of V_3^3; those on the generators of the cone of apparent contour project into the points of the quartic surface of branch points $L_4(x) \equiv u_2^2(x) - u_1(x) \cdot u_3(x) = 0$ in which $u_1 = 0$ is the tangent S_3 to V_3^3 at O_x. The plane

$u_1=0$ in π touches $L_4(x)=0$ along the conic $u_2=0$, which contains the six double points on $L_4(x)=0$, defined by $u_1=u_2=u_3=0$. Between the two three-spaces π and (y) there is a (1, 2) correspondence, in which $L_4(x)=0$ in π is the surface of branch points.

Let the equation of V_3^3 be $u_1x_5^2+2u_2x_5+u_3=0$. Then the projection of V_3^3 upon $x_5=0$ as double S_3 from $O_x\equiv(0, 0, 0, 0, 1)$ has the surface $L_4(x)=0$ of branch points. Let O_y be the image of O_x supposed to be non-singular in $\rho x_i=f_i(y)$. Then (y_0) satisfy the four equations $f_i(y)=0$, $(i=1, 2, 3, 4)$.

In the (1, 2) correspondence between (y) and $x_5=0$, the point O_5 is singular. In the projection upon $x_5=0$ from O_x, the conjugate of O_x is the plane $u_1=0$; it is the intersection with $x_1=0$ of the tangent S_3 to V_3^3 at O_x. The conjugate of O_y is then the f of the ∞^4 system defined by $u_1(f)=0$.

The points of $u_1=0$ in π are of three kinds, those not on L_4 are all conjugate to O_x; those on $u_2=0$, $u_1=0$ but not on $u_3=0$ are all branch points at O_x, as $u_2=0$ defines the quadric cone of inflexional tangents to V_3 at O_x.

Finally, the six double points $u_1=0$, $u_2=0$, $u_3=0$ are images in $x_5=0$ of the six lines of V_3 which pass through O_x.

Any S_3 passing through O_x meets V_3 in a cubic surface which is transformed into itself by interchanging the two points of V_3 on every line through O_x.

Hence in (y), each $f_i(i\neq 5)$ is transformed into itself. The curves $f_k=0$, $f_i=0$, $(i, k\neq 5)$ of genus 1 are the images of the lines of S_3.

The ∞^4 system $|f|$ is not invariant under I_2, and the subsystem is still of grade 3 when the fixed point O_y is adjoined.

A section $\sum a_ix_i=0$ not passing through O_x meets V in a general cubic surface. The three-dimensional cone joining the surface to O_x meets V in a sextic surface, the complete intersection of V and a quadratic variety $|2x|$, obtained from $\sum a_ix_i$ by the quadratic transformation

$$(T) \qquad \begin{cases} \rho x_i' = u_1x_i, & (i=1, \cdots, 4), \\ \rho x_5' = -u_1x_5-u_2. \end{cases}$$

The corresponding transformation in (y) becomes

$$\sigma f_i(y_1', \cdots, y_4') = u_1(f(y))f_i(y),$$
$$\sigma f_5(y_1', \cdots, y_4') = - u_1(f(y))f_5(y) - u_2(f(y)),$$

and for the involution I_2, of associated points,

$$f_i(y') = f_i(y).$$

Thus, the quadratic image of any S_3 section in the transformation T passes through the quadric surface $u_1=0$, $u_2=0$, which is the intersection of the tangent space to V_3 at O_x and the quadric polar of O_x as to V_3.

This is the quadric of inflexional tangents to V_3 at O_x. It meets V_3 in the six lines passing through O_x.

The images of these six lines are six curves through O_y, common to the conjugates of the $\infty^4 |f|$. They are all parasitic curves.

Suppose V_3^3 has a double point not lying in the tangent S_3 at the center of projection. Then in $x_5=0$ the surface of branch points $L_4=0$ has a double point not lying in the singular plane $u_1=0$ associated with the center of projection.

Conversely, if $L_4=0$ has a double point not lying in $u_1=0$, from the vanishing of the first derivatives of L_4 as to x_1, those of V_3^3 can be shown to vanish. Hence we may draw the following conclusion.

A necessary and sufficient condition that the surface of branch points $L_4=0$ has a double point not in the singular plane associated with the center of projection is that V_3^3 has a double point not in the tangent space at the center of projection.

If the values x_i are substituted in L_4, the result is a perfect square $K^2(y)=0$: the surface of contact of V and its three-dimensional tangent cone from O_x.

If V_3^3 has a double point P not at O_x, its image will be an actual double point of $L_4=0$, not on $u_1=0$. In this case, the parametric representation of $V_3^3:P^2$ can be obtained immediately by projecting V_3^3 from P and cutting the projecting cone by any S_3 not passing through P.

If $u_1 \equiv x_1$, and if u_2 and u_3 contain x_1 at most to the first power, then

$$V_3 \equiv u_1 x_5^2 + 2u_2 x_5 + u_3$$
$$= x_1 x_5^2 + 2(x_1 v_1 + w_2) x_5 + w_3 = 0 \equiv x_1 t_2 + t_3,$$

where v_i, w_i are ternary in x_2, x_3, x_4 of degree i, and

$$t_2 = x_5^2 + 2v_1 x_5 + v_2, \qquad t_3 = 2w_2 x_5 + w_3,$$

represents a cubic variety having a double point at the point (1, 0, 0, 0, 0).

The representation in the space (y) now takes the form

$$\rho x_1 = -t_3(y_2, \cdots, y_5),$$
$$\rho x_i = y_i t_2(y_2, \cdots, y_5), \qquad (i = 2, \cdots, 5).$$

The image of the double point is the quadric surface $t_2 = 0$, and the fundamental element in (y) is the sextic space curve γ_6 of genus 4, $t_2 = 0$, $t_3 = 0$.

The ∞^4 linear system, images of the S_3 sections of V_3, now has the form

$$\sum a_i y_i t_2 - a_1 t_3 = 0;$$

thus it includes all the cubic surfaces passing through γ_6. This system satisfies conditions (a), (b) and (c).

The image in (y) of the vertex $(0, 0, 0, 0, 1) \equiv O_x$ is the point (0, 0, 0, 1), hence the ∞^3 subsystem which defines the (1, 2) correspondence between $x_5 = 0$ and (y) has the form

$$\rho x_1 = -2w_2(y_2, y_3, y_4) y_5 + w_3(y_2, y_3, y_4),$$
$$\rho x_2 = y_2 t_2(y_2, \cdots, y_5),$$
$$\rho x_3 = y_3 t_2(y_2, \cdots, y_5),$$
$$\rho x_4 = y_4 t_2(y_2, \cdots, y_5),$$

$$\sigma y_2 = x_1 x_2,$$
$$\sigma y_3 = x_1 x_3,$$
$$\sigma y_4 = x_1 x_4,$$
$$\sigma y_5 = -(v_1 x_1 + w_2) \pm \left[(v_1 x_2 + w_2)^2 - x_1(x_1 v_2 + w_3)\right]^{1/2}.$$

The equations of I_2 in (y) can now be determined at once. The transformation is quartic, monoidal perspective. The surface of invariant points is

$$w_2 y_5^2 - w_3 y_5 + v_1 w_3 + v_2 w_2 = 0.$$

Now suppose that V_3^3 is always rational, and that its equation contains a parameter λ, such that when $\lambda = 0$ the corresponding variety acquires a double point at P. Then

$$\rho x_i = \phi_i(y,\lambda) = y_i v_2(y) F(y) + \lambda f_i(y,\lambda), \quad (i = 2, \cdots, 5),$$
$$\rho x_1 = \phi_1(y,\lambda) = -t_3(y)F(y) + \lambda f(y,\lambda).$$

Consider any linear system of at least ∞^4 rational surfaces of order N. By using the common basis elements as part or all of the fundamental elements of the transforming system, suppose the system has been reduced to its simplest form. Then within the system construct a subsystem which satisfies conditions (a), (b), (c). Evidently, by any birational transformation of (y), the properties of $V(x)$ will remain unchanged. Any rational surface may be transformed by means of its adjoints into one having plane sections of genus 0, 1, 2, 3 or of any positive genus, but hyperelliptic. (Enriques (33), (34).) We shall consider only those linear systems which belong to one or another of these five types, I, $\cdots$, V and which satisfy the conditions (a), (b), (c).

Finally, we select an additional basis point to define the (1, 2) correspondence. The five types will be considered in turn, but the three following properties will first be established.

9. *Properties of Basis Elements.* In the ∞^4 system $|f(y)|$, images of the spatial sections $|x|$ of V_3^3, is a subsystem ∞^3 $|\bar{f}(y)|$ having an additional basis point not a basis point of the larger ∞^4 system. This special basis point is the image of an ordinary point O_x on V_3^3. If now V_3^3 be projected upon a space π not passing through O_x from O_x doubly, then between π and (y) exists a (1, 2) correspondence, such that each $\bar{f}(y)$ is transformed into

itself, both images of a point in the image plane in π being in this surface. In the projection of V_3^3 from O_x upon π, the fundamental element is the plane, which is the intersection of the tangent S_3 to V_3^3 at O_x with π. The image of this plane in (y) is a definite $\bar{f}$ of the subsystem.

If $|\bar{f}|$, and therefore $|f|$, has any other simple basis point, there is one surface $\bar{f}$ belonging to it having this point for double point; the point and the nodal surface are conjugate in the associated I_2 of (y). The associated plane in π must therefore touch L at every common point, since its image in (y) is composite; that is, the plane is a singular tangent plane of L and hence contains six double points. In no cases can all these points be in the first singular tangent plane. Therefore we have the following results.

(A) *If $|f|$ contains a simple isolated basis point, not that defining $|\bar{f}|$, L has double points not in the characteristic singular plane.*

Similarly, if any surface of the system $|\bar{f}|$ is composite, the two components are conjugate to each other in I_2, and the associated plane in π must touch L at every common point; that is, it is a singular plane; hence L contains double points not in the singular plane u_1. (Sharpe and Snyder (100).)

(B) *If $|\bar{f}|$ contains a composite surface, L has double points not in the characteristic singular plane.*

Let δ be a curve in (y), which meets the surfaces of $|f|$ only in basis points or in points of basis curves. The surfaces of the system which pass through a point of δ must contain the whole curve. Hence ∞^2 surfaces of $|\bar{f}|$ contain δ. The images of these surfaces are ∞^2 planes having the image of δ in common, hence it must be a point. The curve must lie entirely on K, the surface of coincident points of (y), and the image point is a double point of L. (Segre (89).) If δ does not pass through O_y, this double point on L can not be in the characteristic singular plane of L.

(C) *If (y) contains a curve δ meeting $|f|$ only in basis points,* *and not passing through the basis point O_y of $|\bar{f}|$, L contains a double point not lying in its characteristic singular plane.*

10. *Rational Plane Sections.* I. When the plane sections of a surface $f(y)=0$ are rational the system $|f|$ consists of ruled surfaces or of Steiner surfaces, that is, quartics with three concurrent double lines. (Picard (85), Guccia (63).) The simplest case satisfying conditions (a), (b), (c) is that of quadrics through 5 points. (Sharpe and Snyder (100).) The surface $L_4=0$ is now the 16-nodal Kummer surface, hence $V_3(x)=0$ is nodal. In case of surfaces of order greater than 2, the variable curve of intersection is always rational since the double curves of each f_i must belong to the basis curve of the system. (Castelnuovo (12), Enriques (30).)

11. *Plane Sections of Genus* 1. II. The simplest case is $N=3$. Conditions (a), (b), (c) are satisfied; in every case $L_4=0$ has a number of extra nodes. (Sharpe and Snyder (100).) Compare the general properties of linear systems of plane curves of general genus. (Segre (88).)

$N=4$ gives a system of quartic surfaces with a double conic and certain other basis elements. But this can be reduced to the preceding one by a quadratic transformation. For $N=5$ the basis elements are a double C_5 with a triple point. The residual intersection of any two surfaces of the system is an elliptic quintic curve; the system is ∞^6 of grade 5. The conditions (a), (b), (c) will be satisfied only by imposing two additional points; hence L_4 will have another singular plane and V_3^3 has nodes.

For N greater than 5, there is no residual curve. (Del Pezzo (26).) For varieties with curve sections of genus 1, see Scorza (86).

12. *Plane Sections of Genus* $p=2$. III. This will be discussed in connection with general hyperelliptic sections. The surface defined by $|C_6|:8P^2$ is the double quadric cone, which can be mapped on a quartic surface with a double line, which is associated with a nodal variety.

13. *Plane Sections of Genus* $p=3$. IV. The non-hyperelliptic case has been discussed by Castelnuovo (9) and by Scorza (87). Those reducible to quartics all have fundamental lines, hence they lead to nodal varieties. (Noether (83).)

Those of the first species, that is, those containing a net of rational curves of order 4, meeting by twos in a single point, can in general not be reduced to quartics.

The only exception is that of the monoids, represented on the plane by $C_4\colon 12P^1$, the 12 basis points all lying on a cubic curve, which is the image of the triple point. Whatever system of basis curves is adjoined to obtain a subsystem satisfying conditions (a), (b), (c), fundamental lines are introduced, hence the associated V_3^3 has one or more double points. The rational quartic curves lie in the planes of the bundle having the triple point for vertex. For a discussion of the involutions which leave invariant a web of monoids, see Snyder (105).

The system $|C_4|\cdot 11P^1$ defines a rational quintic surface with a double cubic curve. All the possible forms of residual basis elements that can satisfy conditions (a), (b), (c) are given in (100); in each L_4, and hence V_3^3, has double points.

14. *Digression. Details of a Quintic Surface.* Case J, Sharpe and Snyder (100), merits further study. The table of characteristics as given is incomplete.

Using the notation there employed, the line h which is the partial image of h' in (x'), with which it is in (1, 1) correspondence, is double on each S_{21} of the web, since h' is double on each S_6' of the system in (x'). Moreover, the curves on H_2, which are the residual images of the points of h', are conics.

The curve β_5 meets H_2 in two points not on γ_3.

The line l joining them is axis of a pencil of planes, each of which cuts from H_2 a parasitic conic.

Two of these have for images in (x') the points of contact of h' and L_4', hence the conics lie on K_{10} and are double on each S_{21} of the web. Each of the six lines $[R_4, K_{10}]=6r$ also belongs to the base of $|S_{21}|$; hence we have

$$S_{21}: \quad \gamma_3{}^8\alpha^4\bar{\alpha}^4\beta_5{}^5 6r\rho_5 h^2 2c_2{}^2,$$

$$K_{10}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5{}^2 6r2c_2.$$

A plane meets its conjugate S_{21} in a composite curve consisting of (S_1, K_{10}) and of a residual curve

$$\delta_{11}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5^3\rho_5 h^2 2c_2.$$

Any two S_{21} may also have x_i curves of order i, parasitic on the web $|S_5|$, and appearing to multiplicity i on $|S_{21}|$. The complete intersection then gives

$$441 = 21+192+16+16+125+6+5+4+16+\sum i^3x_i$$

and the intersection with K_{10} furnishes

$$210 = 10+96+8+8+50+6+8+\sum i^2x_i.$$

The only possible solution is $x_1=8$, $x_2=4$. The 8 lines are divided into two groups of four each, bisecants of β_5 meeting γ_3 and α or $\bar{\alpha}$. Hence we may now write

$$S_{21}: \quad \gamma_3{}^8\alpha^4\bar{\alpha}^4\beta_5{}^5 6r\rho_5 h^2 2c_2{}^2 8u_1 4u_2{}^2,$$

$$K_{10}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5\, 6r2c_2 8u_1 4u_2,$$

$$\delta_{11}: \quad \gamma_3^4\alpha^2\bar{\alpha}^2\beta_5{}^3\rho_5 h^2 2c_2 4u_2,$$

where the interpretation of the symbols in the last equation undergoes an obvious modification.

The genus of δ_{11} is consequently 9, and it intersects K_{10} in 12 variable points. The images of these 12 points are the points of contact of δ' and L'; hence δ', the variable double curve on S_6', image in (x') of S_{21}, is of order 6. The genus of δ' is 2.

The curve δ_{11} has a double point D on h, and the plane of δ_{11} meets the image conic in two points P_1 and P_2. The points D and P_1 form one pair of conjugates, and D and P_2 another; hence δ_6' has a double point where it meets h', the tangents at which are distinct from each other and from h'. The point is a triple point on S_6'. The surface S_6' has a composite double curve consisting of δ_6' and of h', the latter passing through a double point on the former. The genus of the composite c_7 can be found as follows. From an arbitrary point can be drawn 8 lines meeting δ_6' twice, accounting for 7 apparent double points, and the line to the double point. The plane determined by the point and h' meets δ_6' in 4 other points, hence $h=8+4=12$, $p=3$.

The web $|S_5|$ can be transformed birationally into a web of monoids. Let γ_3 be defined by

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_3}{x_4}\,.$$

Put $Q_1=x_1x_3-x_2^2$, $Q_2=x_2x_4-x_3^2$, $Q_3=x_1x_4-x_2x_3$. Consider the transformation defined by

$$y_i = Q_ix_4,\ (i = 1,2,3),\ y_4 = \sum u_iQ_i,$$

where u_i is linear in x_1, x_2, x_3. The equations define a (3, 3) non-involutorial transformation between (x) and (y) in which the fundamental elements in (x) are γ_3 and a plane cubic C_3: $x_4=0$, $\sum u_iQ_i=0$ meeting γ_3 in three points. The image in (y) of the plane $x_4=0$ is the point $(0, 0, 0, 1)\equiv O$. Every point of γ_3 is transformed into a straight line, the locus of which is a ruled surface R_5 of order 5, trisecants of an elliptic C_6: O^3. The image of a point of C_3 is a line through O, the locus of which is the cubic cone Γ_3: C_6.

The image of an arbitrary line of (x) is a cubic curve through O; a secant of γ_3 has for image a conic through O and a generator of R_5; a bisecant of γ_3 goes into a line through O and two generators of R_5. Thus the congruence of bisecants of γ_3 is transformed into the bundle O. Since every line of the congruence meets S_5 in one point not on γ_3, the image must be a monoid in (y). The complete image of S_5 is of order 15, but R_5^2 is a component; α^2, $\bar{\alpha}^2$ go into double basis lines through O.

The curve β_5 has for image a composite curve of order 15, consisting of a proper curve β_7 of order 7 and of eight lines on R_5. Since β_5 meets $x_4=0$ in 5 points, β_7 has O for five fold point. The residual $\bar{C}_6$ is simple,

$$s_5\colon O^4\alpha\bar{\alpha}\beta_7\bar{C}_6.$$

The variable curve of intersection is a C_{10} having O^6.

Since the variable C_6 meets each α in two points, the image C_{10} meets each image $\bar{\alpha}$ in two points, hence the curve lies on a quartic cone of genus 1, with two fixed double generators, $[\beta_7, \bar{C}_6]=6$.

15. *Resumption of the Case $p=3$.* This completes the enumeration of webs of rational surfaces with plane sections of genus 3, belonging to the first species. (Castelnuovo (9).) The normal surfaces are of order $16-\kappa$, lying in space of $14-\kappa$ dimensions ($\kappa<12$). Those that are not normal can be projected into one or another of the types just considered. (Clebsch (19), Cremona (24), Sturm (108).)

The F_6 with a double curve of order 7, genus 3, has an actual triple point on both curve and surface. (Bordiga (7), Veronese (110).)

The system $|C_4|:10P^1$ defines a sextic surface having a double curve C_7, $p=3$ with a triple point. The residual curve of order 8 is rational, hence condition (a) is not satisfied. Similarly for surfaces defined by $|C_4|$ having fewer than 10 basis points.

The second species is composed of those surfaces with plane sections of genus 3 which also contain an ∞^2 system of curves of genus 1; this system can be reduced to quartics. (Castelnuovo (9).) If the system of elliptic quartics is composed of plane curves, their planes pass through a point, and the surfaces are quartics having the given point for tacnode.

In every case the surface contains an $\infty^2 I_2$ of points, the lines joining pairs of conjugates being concurrent. From this point the surface can always be projected into a surface of Veronese in S_5, counted twice. The surface is always rational except when the arithmetic genus is -1 and when the curve of invariant points of the g_2^1 from O consists of four concurrent conics. These cases do not concern us. If the curves of genus 1 lying on the surface are space quartics, the surface can be reduced to a quintic of S_3 having three concurrent double lines, and hence a triple point. The net of quartics is cut from the surface by the net of quadric cones passing simply through the double lines.

The F_8 of S_6, represented by $|C_6|:7P^2$, can be projected into an F_4 of S_3 by choosing the four centers of projection on the same elliptic curve lying on it. The result is a quartic of S_3 with a tacnode and four simple concurrent lines in the tangent plane at the tacnode. (Noether (80), Cremona (24).)

The system of curves $|C_6|:7P^2 3P^1$ defines an $F_5:3C_1^{\,2}3C_1^{\,1}$, the three double lines being concurrent, and the three simple lines lying in the planes containing a pair of double lines. In the plane representation, let the double points be A, B, C, D, E, F, G and the simple ones P, Q, R.

The plane images of the three double lines are $C_3: A \cdots GPQ$; $C_3: A \cdots GPR$; $C_3: A \cdots GQR$. From the complete intersection of two surfaces of the system (F_5) the three double lines should be subtracted, hence the plane sections are represented by $|C_{12}|: A^4 \cdots G^4 PQR$. Thus this characteristic is of grade 29, genus 13, dimension 17. Hence there are nineteen linearly independent surfaces of the system.

In order to secure a subsystem satisfying conditions (a), (b), (c), let (F_5) have an additional basis curve C_m, of order m, genus p, meeting the totality of the double lines in S points; then

$$11m - 2p + 2 - 5s = 26,$$
$$5m - p + 1 - 2s = 14,$$

so that $s = m+2$, $p = 3m - 17$. (Sharpe and Snyder (100), p. 71.) The residual intersection of genus 1 is met by any third surface in three points, hence it can be reduced to a cubic through six basis points. It has one of the four forms

(α) C_3: $A\ B\ C\ D\ E\ F$,

(β) C_3: $A\ B\ C\ D\ E\ P$,

(γ) C_3: $A\ B\ C\ D\ P\ Q$,

(δ) C_3: $A\ B\ C\ P\ Q\ R$.

In (α), the residual basis curve C_m is

$$C_9: \qquad A^3 \cdots F^3G^4PQR.$$

The corresponding basis curve on the surfaces $|f(y)|$ is C_7, $p=4$, meeting each double line in 3 points. Through O can be drawn two other bisecants of C_7; these lines are fundamental, having five fixed points of intersection with each $f(y)$ of the system. Hence $L_4(x)$ has additional double points, and consequently V has a double point.

(β). The basis curve is now of order 6, genus 1, meeting two of the double lines each in 3 points and the third in 2 points. Hence as before, two additional bisecants can be drawn from O; these lines are fundamental, giving rise to double points on $L_4(x)$, and hence on V.

(γ). The basis curve is now composite, consisting of two conics which do not intersect, each meeting all the double lines once, and of a line skew to both conics, meeting one double line. The additional secant of both conics from O accounts for double points on V.

(δ). This case is impossible, as the residual basis curve is of order 4 and genus -5.

There is one more possible case of systems $|f(y)|$ where plane sections of genus 3 exists, namely, that having a plane representation of the form $|C_7|$; $7P^2 2P_1$. (Caporali (8).)

The double basis curve of the system $|f(y)|$ consists of a line C_1 and of a sextic γ_6, $p=2$, having two double points on the line. One surface of the system $|f(y)|$ must be composite, hence L_4 and also V_3 have double points.

The cases of systems of species 3 and of species 4, Castelnuovo (9), reduce to systems of quartics with point singularities more complicated than in that of type 2, but without basis lines. (Noether (83).)

The first surface is represented by $|C_6|:7P^2 4P^1$, all 11 basis points lying on a cubic curve γ. Let the singular point be (0, 0, 0, 1), and the tangent plane be $x_1=0$. Two surfaces of the system meet in a curve C_{16} of order 16, with 8 branches through O, all lying in $x_1=0$. The image of a plane section through O consists of γ and of another cubic with the symbol C_3: $ABCDEFG$. The plane representation of C_{16} is $C_{12}:7P^4$. When this is composite, one component being of genus 1 and variable, it can be reduced to the $C_3:A \cdots F$, and the fixed basis curve has for image $C_9:A^3 \cdots F^3G^4$, $p=4$. In space the variable curve is a C_6 with three branches through O; the basis curve is then C_{10} with five branches through O. Since two trisecants of C_{10} can be drawn through O, these lines are fundamental, giving rise to double points on V_3^3.

It was assumed that the variable cubic in the plane representation had for basis points A, B, C, D, E, F; if instead one or more be chosen from the simple basis points of the system, practically the same argument may be repeated, with the same conclusion.

Another case is that of F_4 represented by $|C_7|:P^39P^2$, all the basis points lying on a cubic curve, and another is the F_4 represented by $|C_8|:8P^3P^2P^1$, the ten basis points lying on a cubic. Both of these can be considered as the preceding case, with the same result.

This completes the case $p=3$, non-hyperelliptic. When the net of curves on each surface of the system is composite, the plane sections of genus 3 are hyperelliptic. In this case the F_{16} is represented by $|C_6|:P^4P^2$ or by $|C_{8-k}|:P^{6-k}(3-k)P^2$, $(k=0, 1, 2, 3)$, the double points being adjacent to the P^{6-k}. They will be considered in connection with the general hyperelliptic case. (De Franchis (25).)

16. *General Hyperelliptic Sections.* V. From Castelnuovo (10) and Enriques (31) we know that every surface with hyperelliptic plane sections or by sections S_{n-1}, if it lies in S_n, is rational. If the genus of the sections exceeds 1, the surface contains a rational ∞^1 system of conics, such that through any point of the surface passes just one conic of the system. If the section is of order n and genus π, the surface cannot belong to a space larger than $S_{n-\pi+1}$; if it belongs to a space of lower dimensionality, it is the projection of a normal surface of $S_{n-\pi+1}$.

The planes of the ∞^1 conics on the surface form a variety of order not larger than $n-\pi-1$.

The surface can be mapped on a plane such that the plane sections (or those of S_{r-1} if in S_r) are represented by curves of order k having a common point of multiplicity $k-2$, and possibly other basis points. Hence if the genus is π, the maximum order is $4\pi+4$. Every such surface of lower order can be obtained from a surface of order $4\pi+4$ of $S_{3\pi+5}$ from $4\pi+4-n$ points upon it, and possibly from other points not on the surface.

In order to obtain a subsystem satisfying conditions (a), (b), (c), it is necessary to impose simple basis points in every case. When double points exist among the additional basis points, the curves can be reduced to those of lower order. Several cases of $|C_k|:P^{k-2}$ have been considered in connection with other problems. Let x be the number of simple basis points, N the order of the surface, m the order of its double curve. Then for (k, x, N, m) (4, 8, 4, 1) see (100), p. 61; (4, 7, 5, 3) see (100), p. 72; (4, 6, 6, 8) see (8), p. 202; (5, 11, 5, 3) see (100), p. 71; (5, 10, 6, 7) see (8), p. 207.

The system $|F_N|:C_1^{N-2}\ \gamma$ can be treated as in (100) p. 62, and the monoidal cases of any order necessarily have fundamental lines.

If the assumption made in § 8 is justifiable it follows that the general cubic variety of S_4 is irrational.

17. *Reduction to Monoidal Types.* From the preceding representation, follows at once the proof of the following theorem.

All (1, 2) *quaternary correspondences of genus* 1, *in which the quartic surface of branch points has a singular plane and at least one double point not in this plane, can be expressed in terms of a* (1, 1) *correspondence between the points of an* S_3' *and the pairs of points of a perspective monoidal involution of order two in another space* S_3.

Let there be a second involution I_2 in (x), having the same surface of branch points L_4' in (x'). Since the points of V_8

and of (x) are in (1, 1) correspondence, we have in any case $y_i = f_i(x_1, x_2, x_3, x_4)$, $i = 1, 2, 3, 4$. For example, by including the vertex P on V_3 from which the variety is projected on the double S_3, we may write

$$y_i = F_2(x)x_i; \quad y_5 = F_3(x).$$

Let the pairs of points of I_2 also be represented on V_3. Since the two surfaces of coincident points K_1 of I_1 and K_2 of I_2 are each in (1, 1) correspondence with L_4', they can be superposed one on the other.

Then I_2 is transformed into a second monoidal involution, having a second double point Q_x on K_2 for vertex. The image on V_3 of Q_x is a second double point, hence L_4' has a second double point. The line joining the double points on V_3 lies entirely upon it; every plane through it meets V_3 in a residual conic, image of a line of the bundle Q_x in (x). This line contains an infinite number of pairs of conjugate points with double points on K_x. Hence the conic on V_3 contains an infinite number of pairs of points forming an involution in which the double points are the residual points of intersection of the plane of the conic with the curve of contact K_V of the tangent cone from P. Thus, by I_2^V every point of K_V remains invariant; by projection upon $x_5' = 0$, the associated point on L_4' is a branch point as before.

If in (x) there exist several involutions having the same quartic surface of branch points with a singular plane, by the above process they can all be reduced to perspective monoidal involutions by the same transformation.

When L_4' has only one double point not in the singular plane, the web of invariant cubic surfaces in (x) has a simple point and a general $C_6, p=4$, for basis elements. It is described as case D of cubics in Sharpe and Snyder (100), p. 58. It does not appear again in a transformed form in the enumeration there given.

If L_4' has two double points not in the singular plane, $K(x)$ has a second double point, the quadric F_2 containing C_6 and the cubics of the web touch each other, C_6 has an actual double point and is now of genus 3.

Similarly, L_4' may have 3 or 4, . . . or 10 double points not in its singular plane. In the latter case it is the 16-nodal Kummer surface, and K may be reduced to the symmetroid.

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