THE EXISTENCE OF THE LEBESGUE-STIELTJES INTEGRAL*

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A definition of a Lebesgue-Stieltjes integral of a function $f(x)$ defined on (a, b) with respect to a non-decreasing function $V(x)$ bounded on (a, b) has been given by Hildebrandt.†

This definition involves the idea of the measurability of f with respect to V. If α is the interval $a'<x<b'$, then $V(\alpha)=V(b'-0)-V(a'+0)$. Let a set E be enclosed in a finite or countably infinite set of non-overlapping open intervals $A\equiv\alpha_1, \alpha_2, \cdots$. Let $V(E)$ be the lower limit of $V(A)=\sum V(\alpha_i)$ for all possible enclosures A. In the same way define $V(CE)$. When

$$V(E)+V(CE)=V(a,b)=V(b)-V(a), \tag{1}$$

the set E is said to be measurable relative to V. If for all real values of l the set for which $f>l$ satisfies (1), then f is measurable relative to V. Hobson‡ gives a definition which involves a different formulation of the same idea. To state this we shall make use of the following correspondence between the points of $\alpha=V(a)\leqq u\leqq V(b)=\beta$ and $a\leqq x\leqq b$. First, if x is a point of discontinuity of V, then x goes by means of $u=V(x)$ into the closed interval $V(x-0)\leqq u \leqq V(x+0)$. There will then correspond to each u on (α, β) at least one value of x on (a, b). If to a value of u there corresponds more than one value of x, then V is constant throughout an interval, and x_u shall be the lower end point of this interval, or the lower bound of points of the interval in case it is open. If $f(x)$ is any function defined on (a, b), then $\psi(u)$ is defined by $\psi(u)=f(x_u)$ and

$$LS\int_a^b f(x)dV(x) = L\int_\alpha^\beta \psi(u)du,$$

* Presented to the Society, September 7, 1928.

† This Bulletin, vol. 24, pp. 188–190.

‡ *Theory of Functions of a Real Variable*, 3d ed., vol. I, §445.

when the latter exists. Thus the requirement that f be measurable relative to V is replaced by the requirement that ψ be measurable on (α, β). In this note we determine some necessary and sufficient conditions bearing on V under which all Lebesgue measurable functions f are measurable relative to V. These conditions are also necessary and sufficient to insure that all measurable functions f are carried into functions $\psi(u)=f(x_u)$ measurable on (α, β).

As a matter of notation, we shall use A and B to indicate sets of intervals enclosing E and CE respectively. Each set shall be non-overlapping and open, and each interval shall contain at least one point of the set enclosed. E_u shall denote the set into which E_x is carried by means of the relation $u=V(x)$, a point of discontinuity of V going into a closed interval as agreed upon above. If $x_1, x_2, \cdots$ are the countable set of discontinuities of V and s_i the saltus of V at x_i, then, for a given ϵ, $K_x \equiv x_1, x_2, \cdots, x_k$ where $\sum_{i=k+1}^{\infty} s_i < \epsilon/2$. For convenient reference we state some properties* of all sets G which are measurable relative to V.

(1) If E_1 and E_2 belong to G then E_1+E_2 belongs to G.

(2) The Borel measurable sets belong to G.

(3) A necessary and sufficient condition that E be measurable relative to V is that there exist intervals A and B enclosing E and CE such that $V(AB)<\epsilon$.

(4) Any set E on a set of open intervals throughout which V is constant is measurable relative to V.

This follows at once from (3). For evidently E can be put in A where $V(A)=0$. If then CE is put in any set B, we have $0 \leqq V(AB) \leqq V(A)$, which shows that (3) is satisfied.

THEOREM I. *A necessary and sufficient condition that E_x be measurable relative to V is that E_u be measurable on (α, β).*

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* (1) and (2) are proved by Radon, Wiener Sitzungberichte, vol. 122, pp. 1305 ff. (3) is proved by Bliss, (this Bulletin, vol. 24, p. 13), for continuous monotone functions, but the proof is applicable to any non-decreasing function.

Suppose E_x measurable relative to V. Then by (3) we can determine A_x and B_x so that $V(A_xB_x)<\epsilon$. The open intervals A_x go, by means of $u=V(x)$, into a set of open intervals A_u together with a countable set of points A_u' which belong to E_u, and E_u is on A_u+A_u'. Similarly CE_u is on open intervals B_u and a countable set of points B_u'. But $m(A_uB_u)=V(A_xB_x)<\epsilon$. This, together with the fact that A_u' and B_u' are countable, shows that E_u is measurable.

Suppose E_u is measurable. Let $\overline{E}_u$ be the closed intervals of E_u which come from points of discontinuity of V. $\overline{E}_x$ is then a countable set of points which, on account of (2) is measurable relative to V. Let $e_u=E_u-\overline{E}_u$. If K_u is the set of closed intervals coming from K_x, then e_u is on the open intervals CK_u. Put e_u in A_u and ce_u in B_u where A_u is on CK_u and where $m(A_uB_u)<\epsilon/2$. The open intervals A_u and B_u correspond to open intervals A_x and B_x which contain e_x and ce_x respectively. And since A_u is not on K_u, A_x contains none of the points of the set K_x. Consequently (A_xB_x) contains none of the set K_x, and as a result of this

$$V(A_xB_x) \leqq m(A_uB_u) + \sum_{i=k+1}^{\infty} s_i < \epsilon.$$

It then follows from (3) that e_x is measurable relative to V, and by (1) so also is $E_x=\overline{E}_x+e_x$.

It has been shown by Carathéodory* that the function V can be represented as the sum of two functions $V=\phi+\chi$, where ϕ depends only on the discontinuities of V and where χ is continuous. We can now prove the following theorem.

THEOREM II.† *In order that every measurable set E_x go into a measurable set E_u, it is necessary and sufficient that χ be absolutely continuous.*

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* *Vorlesungen über Reelle Funktionen*, §153.

† The corresponding theorem for continuous functions has been proved by Rademacher, (Monatshefte für Mathematik und Physik, vol. 27 (1916), p. 266). His theorem is the special case of ours where ϕ is identically zero.

The condition is sufficient. Suppose E_x measurable. Let ${}_1E_x$ be the part of E_x which belongs either to the discontinuities of V or to intervals throughout which V is constant. ${}_1E_u$, being a set of closed intervals and a countable set of points, is measurable. We complete the argument by showing that $e_u = E_u - {}_1E_u$ can be put in A_u and ce_u in B_u, where $m(A_uB_u) < \epsilon$. The function χ being absolutely continuous, there exists a $\delta > 0$ such that if D is any set of intervals with $m(D) < \delta$, then $\chi(D) < \epsilon/2$. The set e_x contains none of the points of K_x. Hence, since e_x is measurable and since K_x is a finite set, e_x can be put in A_x and ce_x in B_x, where A_x contains none of the points of K_x, and where $m(A_xB_x) < \epsilon/2$. Since each interval of the set A_x contains at least one point of e_x, and since the end points of B_x are points of e_x it follows that A_x and B_x correspond to open intervals A_u and B_u containing e_u and ce_u respectively. Again, since A_x contains none of the points of K_x, neither does (A_xB_x). Hence,

$$m(A_uB_u) = V(A_xB_x) = \phi(A_xB_x) + \chi(A_xB_x) < \sum_{i=k+1}^{\infty} s_i + \epsilon/2 < \epsilon.$$

The condition is also necessary. Suppose χ is not absolutely continuous. For a given $\epsilon > 0$ let $\chi(\delta)$ be the upper limit of $\chi(\alpha)$ for all possible sets of intervals α for which $m\alpha < \delta$. Obviously, if $\delta_1 < \delta_2$, then $\chi(\delta_1) \leqq \chi(\delta_2)$. And since χ is not absolutely continuous, $\lim_{\delta \to 0}\chi(\delta) = d > 0$. Making use of these facts we now prove the following lemma.

LEMMA I. *There exists on (a, b) a set of points D_x of zero measure which contains no points of discontinuity of V, and no points of intervals throughout which V is constant, and which is carried, by means of $u = V(x)$, into a measurable* set D_u with $mD_u \geqq d$.*

Choose $\delta_1, \delta_2, \cdots$, an infinite sequence of positive numbers approaching zero monotonically, and such that $\sum \delta_i$

* Carathéodory (loc. cit., §512) has considered the analogous problem for continuous monotone functions. By a method different from ours he arrives at the existence of a set D_x of zero measure which goes into a set D_u with outer measure $\geqq d$.

converges. Since $\chi(\delta_i) \geqq d$ for all i, we can find a set of intervals α_x^i such that $\chi(\alpha_x^i) > d - \epsilon$, and $m\alpha_x^i < \delta_i$. The infinite sequence of sets of intervals α_x^i defines, by means of $u = \chi(x)$, an infinite sequence of sets of intervals α_u^i on $\alpha' = \chi(a) \leqq u \leqq \beta = \chi(b)$, where $m\alpha_u^i = \chi(\alpha_x^i) > d - \epsilon$, $(i = 1, 2, \cdots)$. It then follows* that on (α', β') we have a measurable set of points D_u''' where $mD_u''' > d - \epsilon$, and each point of which belongs to an infinite number of the sets of intervals α_u^i. If we remove from D_u''' the countable set which comes by means of $u = \chi(x)$ from the countable set of discontinuities of V, there remains a set D_u'' which still has measure $> d - \epsilon$. Let D_x'' be the set on (a, b) whose image, by means of $u = \chi(x)$, is D_u''. Each point of D_u' is on an infinite set of the sets of intervals α_u^i. Consequently each point of the set D_x'' is on an infinite set of the sets of intervals α_x^i. Thus, for $i = l$, where l is as large as we wish, D_x'' is on an infinite set of the intervals $\alpha_x^{l+1}, \alpha_x^{l+2}, \cdots$. Hence $mD_x'' \leqq \sum_{i=l+1}^{\infty} m\alpha_x^i \leqq \sum_{i=l+1}^{\infty} \delta_i$. On account of the convergence of $\sum \delta_i$ and the fact that we can take l as great as we please, it follows that $mD_x'' = 0$. If now we consider a sequence of values of ϵ approaching zero, we get a sequence of sets ${}_1D_x''$, ${}_2D_x''$, $\cdots$, where ${}_iD_x''$ has zero measure and ${}_iD_u''$ has measure $\geqq d - \epsilon_i$. It follows that the set D_u' which consists of all the points in any of the sets ${}_iD_u'$ has measure $\geqq d$, and the corresponding set D_x' has measure zero.

It will now be shown that $u = V(x)$ carries D_x' into a set $\overline{D}_u$ measurable on (α, β) with $m\overline{D}_u \geqq d$. On (α', β'), D_u' is measurable and contains none of the points K_u which come from K_x by means of $u = \chi(x)$. Hence D_u' can be put in A_u and CD_u' in B_u where A_u contains none of the finite set K_u, and where $m(A_u B_u) < \epsilon/2$. The intervals A_u and B_u correspond by means of $u = \chi(x)$ to intervals A_x and B_x, where A_x contains D_x' and contains none of the points of K_x,

* W. H. Young, Proceedings of the London Mathematical Society, (2), vol. 2, p. 26. Also Borel, Comptes Rendus, December, 1903.

and where $\chi(A_xB_x)=m(A_uB_u)\leqq\epsilon/2$. But since A_x contains no points of K_x neither does (A_xB_x). Hence

$$V(A_xB_x) = \phi(A_xB_x) + \chi(A_xB_x) \leqq \sum_{i=k+1}^{\infty} s_i + \frac{\epsilon}{2} < \epsilon.$$

Hence D_x' is measurable relative to V. It then follows from Theorem I that $\overline{D}_u$ is measurable on (α, β). And evidently $m\overline{D}_u$ cannot be less than $\chi(D_x')\geqq d$. If we now remove from $\overline{D}_u$ the countable set which comes by means of $u=V(x)$ from intervals throughout which V is constant, we arrive at a set D_u, where $mD_u\geqq d$, and where the corresponding set D_x contains no discontinuities of V and no points of intervals throughout which V is constant, and where $mD_x=0$.

If N_u is any non-measurable component of D_u, then N_x is part of D_x and $mN_x=0$. But $u=V(x)$ carries N_x into N_u. Hence V does not carry every set measurable on (a, b) into a set measurable on (α, β). If $f(x)$ is a function such that $f=1$ on N_x and zero elsewhere on (a, b), then $\psi(u)=f(x_u)=1$ on N_u and zero elsewhere on (α, β). Thus $\psi(u)$ is not measurable on (α, β). Also, by Theorem I, f is not measurable relative to V. We have the following result.

THEOREM III. *A necessary and sufficient condition that every function f measurable on (a, b) be measurable relative to V is that χ be absolutely continuous. This condition is also necessary and sufficient that $\psi(u)=f(x_u)$ be measurable on (α, β).*

THEOREM IV.* *A necessary and sufficient condition that every function f bounded and measurable on (a, b) possess a Lebesgue-Stieltjes integral in the sense of Hildebrandt, or in the sense of Hobson, is that χ be absolutely continuous.*

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* M. D. Menchoff and Mlle. Bary, Annali di Mathematica, (4), vol. 5, have considered the Lebesgue-Stieltjes integral of f with respect to V where V is absolutely continuous. They show (p. 24) that the integral according to their definition exists whenever f is bounded and measurable. Their discussion is only applicable to the case where V is absolutely continuous.