NOTE ON THE PLÜCKER EQUATIONS FOR PLANE ALGEBRAIC CURVES IN THE GALOIS FIELDS

BY A. D. CAMPBELL

In this note we shall consider the Plücker* equations for plane algebraic curves and find out in how far they are valid in the Galois fields of order p^n. We designate these fields by $GF(p^n)$. Let us consider the n-ic

$$
\begin{aligned}
f(x,y,z) \equiv\ & az^n + (b_0x + b_1y)z^{n-1} \\
& + (c_0x^2 + c_1xy + c_2y^2)z^{n-2} \\
& + (d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3)z^{n-3} + \cdots \\
& + (l_0x^n + \cdots + l_ny^n) = 0,
\end{aligned} \tag{1}
$$

the first polar† of (0, 0, 1) with respect to (1),

$$
\begin{aligned}
naz^{n-1} &+ (n-1)(b_0x + b_1y)z^{n-2} \\
&+ (n-2)(c_0x^2 + c_1xy + c_2y^2)z^{n-3} + \cdots = 0,
\end{aligned} \tag{2}
$$

and the polar conic of (0, 0, 1) with respect to (1),

$$
\frac{n(n-1)}{2}az^2 + (n-1)(b_0x + b_1y)z + (c_0x^2 + c_1xy + c_2y^2) = 0. \tag{3}
$$

In the derivation of the Plücker equations given by Hilton‡, use is made of the number of intersections of the n-ic and its first polar and of the n-ic and its Hessian, also the polar reciprocal of the n-ic with respect to a given conic is utilized. If $n = \alpha p^r + 1$ the polar conic (3) passes through (0, 0, 1) and is degenerate, but (0, 0, 1) can be any point P, hence for these $GF(p^n)$ we cannot use the Hessian in the derivation of any of Plücker's equations. Also in the $GF(2^n)$ the polars of all the points in the plane with respect to any given conic

* See Hilton, *Plane Algebraic Curves*, p. 112.

† See A. D. Campbell, *The polar curves of plane algebraic curves in the Galois fields*, this Bulletin, vol. 34 (1928), pp. 361–363.

‡ Loc. cit., pp. 97, 100, 63, 34, 66.

are concurrent (as is easy to prove), hence in these fields we cannot use the polar reciprocal of the n-ic with respect to a conic.

We get around these difficulties in the following manner. Hilton shows* that if we can establish the three following equations, the others follow from these three:

$$m = n(n-1) - 2\delta - 3\kappa, \tag{4}$$

$$n = m(m-1) - 2\tau - 3\iota, \tag{5}$$

$$\tfrac{1}{2}n(n+3) - \delta - 2\kappa = \tfrac{1}{2}m(m+3) - \tau - 2\iota, \tag{6}$$

where n is the order of the curve under consideration, m is its class, δ the number of nodes on the curve, κ the number of cusps, τ the number of bitangents, ι the number of inflections. Just as in Hilton pp. 96 and 97† so here we can readily see that (2) cuts (1) once at the point of contact of each tangent from (0, 0, 1) to the given curve, twice at each node and thrice at each cusp of the given curve. Also the curves (1) and (2) ordinarily intersect in $n(n-1)$ points, hence equation (4) is in general valid in the $GF(p^n)$. In some cases (1) and (2) do not intersect in $n(n-1)$ points, namely when (2) is composite with a repeated factor. Such a case occurs for $p=2$, $n=3$, and (1) lacking an xyz term.

The equations (5) and (6) are established in Hilton by taking the polar reciprocal of the n-ic with respect to a conic and by the use of Sylvester's dialytic method.‡ The fact is used (p. 63) that from a point on the tangent t at a cusp P only one tangent to the n-ic coincides with t, while at P three tangents coincide with t. This fact is necessary in order to show that the polar reciprocal of a cusp is a point of inflection. The proof of this fact that is given in Hilton (pp. 84, 86) is not applicable to the $GF(p^n)$, so we substitute the following proof. If we take P as (0, 0, 1) with $y=0$ as tangent,

* Loc. cit., p. 112.

† Using the eliminant as in Hilton, pp. 10, 11.

‡ Loc. cit., pp. 97, 63, 66.

then in (1) $a=b_0=b_1=c_0=c_1=0$ and $c_2\neq 0$, hence the curve* and the first polar of P have in common only $n(n-1)-2\cdot 2-2$ points beside P, whereas for P' an ordinary point the curve and the first polar of P' have in common $n(n-1)-3$ points beside the cusp P. From this follows the above fact. In the $GF(p^n)$ for $p>2$ we can take the polar reciprocal of an n-ic with respect to a conic. Also we do not need to use the Hessian because (5) and (6) are proved in Hilton by means of the polar reciprocal of the n-ic;† and from (5) and (6) we can obtain what Hilton obtains from the intersections of the n-ic and its Hessian, namely the equation‡

$$\iota = 3n(n-2) - 6\delta - 8\kappa. \tag{7}$$

Therefore the Plücker equations are valid in the $GF(p^n)$ for $p>2$. In the $GF(2^n)$ we cannot use a polar reciprocal of the n-ic; we can, however, use plane duality§ in these $GF(2^n)$ to replace the use of the polar reciprocal in proving (5) and (6). Therefore also in the $GF(2^n)$ the Plücker equations are still valid.

Syracuse University

---

* Hilton, loc. cit., p. 11.

† Loc. cit., pp. 33, 66.

‡ Loc. cit., pp. 100, 101.

§ See Veblen and Young, *Projective Geometry*, vol. 1, p. 201.