BELL ON ALGEBRAIC ARITHMETIC

Algebraic Arithmetic. By E. T. Bell. Colloquium Publications of the American Mathematical Society, volume 7, 1927. 180 pp.

This book of marked originality is of vital interest to advanced students in various branches of mathematics, including the theory of numbers, abstract algebra, elliptic and theta functions, Bernoullian numbers and functions, and the foundations of mathematics.

A central feature is the new presentation of the author's principle of arithmetical paraphrases, which won him the Bôcher prize in 1924, jointly with Professor Lefschetz. This general principle serves to unify and extend many isolated results in the theory of numbers. In a modified form, this principle may be stated very simply for a special, but typical, case. Suppose that by two developments of an elliptic or theta function we have found a linear identity between the cosines of the angles a_ix+b_iy for $i=1, 2, \cdots$; then if $f(a,b)$ is any function which is unaltered by the change of a to $-a$ or of b to $-b$, we have the same linear identity between the $f(a_i,b_i)$. Since we are free to choose the even function f, we obtain an infinitude of arithmetic facts.

The general principle of paraphrase (pp. 66–68) starts with a linear identity between terms each a product of r cosines and s sines and leads to the same identity between values of any function which is even in r variables and odd in s. It holds also in the extended sense that the variables are any one-rowed matrices. Several illustrations are given to show in detail how the initial identities are obtained from theta functions of one or more arguments. The applications include many classic problems in the theory of numbers.

A leading feature of the book seems to the reviewer to be its success in a systematic attempt to find a unified theory for each of various classes of related important problems in the theory of numbers, including its interrelations with algebra and analysis. An older case was the development of a theory of abstract groups which includes as special cases the really vital results and processes of the various theories of concrete groups, such as groups of permutations of letters, groups of linear or other substitutions, and groups of motions. The gain is not merely in great economy, but in the clarification which results from the exclusion of the irrelevant and the concentration on the relevant. Here it was clearly necessary to proceed abstractly. To the immature mind, abstract means obscure. To anyone really initiated in modern mathematics, an abstract theory is the really clear one, since irrelevant facts have been discarded, and the vital facts and processes alone are retained. As between an abstract theory and a concrete one, the former is far more likely to be clear and sound, since its very nature requires that the assumptions be explicitly stated (as postulates) and that the deductions be made by logical processes and without lapse into intuition, consciously or not.

It is therefore wholly to its credit that Bell's book is so largely abstract. In this way he has secured the utmost generality and hence insured permanency to his work. There is also the following important immediate gain. After constructing abstract theories for various classes of problems, he was in a position to decide if those theories are distinct or are (abstractly) identical. In the latter case we have not merely the often astonishing conclusion that two quite different theories are really equivalent, but the great advantage of being able to apply in a particular case whichever of the two theories is best adapted to it.

The book is however not abstract for the sake of being abstract, but because it gives the essence of a vast array of concrete results traced to their true sources and easily deduced from these few sources. It is not to be overlooked that the author has a minute first-hand acquaintance with the vast array of concrete facts in the theory of numbers. It would require many volumes to expound them and the result would bewilder the reader. How much better to have a brief book which epitomizes all these facts under a few abstract theories. This original and scholarly book is an honor to American mathematics.

L. E. DICKSON

FORSYTH ON CALCULUS OF VARIATIONS

Calculus of Variations. By A. R. Forsyth. Cambridge University Press, 1927. xxii+656 pp.

Weierstrass made three very important contributions to the theory of the calculus of variations. Earlier writers had deduced necessary conditions on a minimizing arc $y=y(x)$ $(x_1\leqq x\leqq x_2)$ by comparing the value which such an arc gives to the integral to be minimized with the value given by neighboring arcs of the form $y=y(x)+\delta y(x)$, where $\delta y=a\eta(x)$. By the use of variations δy of another type, Weierstrass deduced a new necessary condition. He also introduced the parametric representation of curves, $x=x(t)$, $y=y(t)$, into the theory. This was important from the standpoint of the older writers because it enables one to vary impartially either of the variables x or y. The effort to do this had caused considerable difficulty and misunderstanding in the earlier theory of the non-parametric case. The parametric theory is even more valuable, however, because it removes the geometric restrictions on the form of curves which are imposed by the non-parametric representation $y=y(x)$, a removal which is necessary for the complete investigation of many geometric problems. Finally Weierstrass formulated clearly the problems which he studied, distinguished clearly between conditions which are necessary for a minimum and those which are sufficient, and devised an ingenious sufficiency proof which under certain circumstances established the minimizing property of an arc $y=y(x)$ as compared with all other neighboring arcs $y=y(x)+\delta y(x)$, irrespective of the form of the variation δy.

In the book here reviewed Professor Forsyth shows that he has been influenced by the first two of the contributions which have just been mentioned as being due to Weierstrass, the so-called Weierstrassian necessary