ON QUANTIFIERS FOR GENERAL PROPOSITIONS*

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General propositions are commonly constructed in terms of the two applicatives "some" and "every." These applicatives occur singly in propositions of the form $(\exists x) \cdot \phi x$, for at least one value of x, ϕx holds, and $(x) \cdot \phi x$, for every value of x, ϕx holds, and in propositions involving more than one variable constituent when these propositions are of the form $(\exists x, \cdots, n) \cdot f(x, \cdots, n)$ or $(x, \cdots, n) \cdot f(x, \cdots, n)$. Whereas, in propositions of the form

$$(\exists x, \cdots, l)\ (y, \cdots, m)\ (\exists z, \cdots, n) \cdots \cdot f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n;\ \cdots)$$

or

$$(x, \cdots, l)\ (\exists y, \cdots, m)\ (z, \cdots, n) \cdots \cdot f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n;\ \cdots),$$

each of the applicatives may have a single or a multiple occurrence.† There are, however, in the traditional treatment of general propositions, four quantitative functions of a property $\phi\hat{x}$, viz., *Every* x *is such that* ϕx, *Some* x *is such that* ϕx, *No* x *is such that* ϕx, and *Not-every* x *is such that* ϕx. These applicatives occur in the formulation of the syllogism in connection with functions of the form $\phi x \supset \psi x$, and they were, it seems, never carried beyond propositions involving a single applicative. In what follows we shall be concerned chiefly to exhibit the formal properties of general propositions and of general propositional functions when they are expressed directly in terms of the quantifiers *no* and *not-every*.

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† All of the functions which occur in the present discussion are first-order functions; their quantified constituents are all variables which denote individuals. This limitation is, however, merely a matter of convenience. The properties of functions to be exhibited are the same whether the quantified constituents denote individuals or functions.

Any function which involves a single occurrence of an applicative may be said to be singly quantified. Thus $(x) \cdot \phi x$ is singly quantified, as is $(\exists x, \cdots, n) \cdot \phi(x, \cdots, n)$. Any function which involves a multiple occurrence of applicatives, the same or different, may be said to be multiply quantified. Thus $(x) : (y) \cdot \phi(x, y)$ is a multiply quantified function, as is $(x)(\exists y)(z) \cdot \phi(x, y, z)$. The first of these functions is doubly quantified, while the second is triply quantified. Functions involving n occurrences of an applicative, the same or different, will be said to be n-tuply quantified. The terminology introduced here is to be understood to apply also to propositions; any value of an n-tuply quantified function is an n-tuply quantified proposition.

$$(x) : (y) \cdot \phi(x,y) : \equiv : (x,y) \cdot \phi(x,y) \text{ and } (\exists x) : (\exists y) \cdot \phi(x,y) : \equiv : (\exists x,y) \cdot \phi(x,y).$$

$$(x, \cdots, m) : (y, \cdots, n) \cdot \phi(x, \cdots, m ; y, \cdots, n) : \equiv : (x, \cdots, m; y, \cdots, n) \cdot \phi(x, \cdots, m ;y, \cdots, n)$$

and

$$(\exists x, \cdots, m)(\exists y, \cdots, n) \cdot \phi(x, \cdots, m ;y, \cdots, n) : \equiv : (\exists x, \cdots, m; y, \cdots, n) \cdot \phi(x, \cdots, m ; y, \cdots, n).$$

Accordingly, when two occurrences of the same applicative are juxtaposed the variables may be combined under the same applicative. This reduces the degree of quantification of the function. In a function no further reducible no two occurrences of the same applicative will be juxtaposed. Such functions will be said to be in reduced form.

$$\sim (x) \cdot \phi x \cdot \equiv \cdot (\exists x) \cdot \sim \phi x \quad \text{and} \quad \sim (\exists x) \cdot \phi x \cdot \equiv \cdot (x) \cdot \sim \phi x,$$

from which it follows that the sign of negation can be removed from before any quantifier.

$$\sim(x, \cdots, l)(\exists y, \cdots, m)(z, \cdots, n) \cdots f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$$

is equivalent to

$$(\exists x, \cdots, l)\ (y, \cdots, m)\ (\exists z, \cdots, n) \cdots \sim f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$$

and, of course,

$$\sim(\exists x, \cdots, l)\ (y, \cdots, m)\ (\exists z, \cdots, n) \cdots f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$$

is equivalent to

$$(x, \cdots, l)\ (\exists y, \cdots, m)\ (z, \cdots, n) \cdots \sim f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots).$$

Accordingly, to get the contradictory of a proposition involving a single complex quantifier in "some" and "every" change every universally quantified constituent of the proposition into a particular and every particularly quantified constituent into a universal and take the negative of the function.

Let "No x is such that ϕx" be denoted by $[x] \cdot \phi x$ and "No $x, \cdots, n$ are such that $\phi(x, \cdots, n)$" by $[x, \cdots, n]$ $\phi(x, \cdots, n)$. Then

$$[x] \cdot \phi x \cdot \equiv \cdot \sim (\exists x) \cdot \phi x \cdot \equiv \cdot (x) \cdot \sim \phi x, \text{ and}$$

$$[x, \cdots, n] \cdot \phi(x, \cdots, n) \cdot \equiv \cdot \sim (\exists x, \cdots, n) \cdot \phi(x, \cdots, n) \cdot \equiv \cdot (x, \cdots, n) \cdot \sim \phi(x, \cdots, n).$$

"Not every x is such that ϕx" may be written $\{x\} \cdot \phi x$, and "Not-every $x, \cdots, n$ are such that $\phi(x, \cdots, n)$," $\{x, \cdots, n\} \cdot \phi(x, \cdots, n)$. Accordingly,

$$\{x\} \cdot \phi x \cdot \equiv \cdot \sim (x) \cdot \phi x \cdot \equiv \cdot (\exists x) \cdot \sim \phi x, \text{ and}$$

$$\{x, \cdots, n\} \cdot \phi(x, \cdots, n) \cdot \equiv \cdot \sim (x, \cdots, n) \cdot \phi(x, \cdots, n) \cdot \equiv \cdot (\exists x, \cdots, n) \cdot \sim \phi(x, \cdots, n).\text{*}$$

Since $(x) \cdot \phi x \cdot \equiv \cdot \sim(\exists x) \cdot \sim\phi x$, any proposition in terms of "some" and "every" can be expressed in terms of "some" and "$\sim$". Thus

$$(x)(\exists y)(z)(\exists w) \cdot f(x, y, z, w) \cdot \equiv \cdot \sim (\exists x)\sim(\exists y)\sim(\exists z) \sim(\exists w) \cdot f(x, y, z, w)$$

* The applicative $\{x\}$ was suggested to me by Dr. H. M. Sheffer, as was the notation used in both cases.

and

$$(\exists x)(y)(\exists z)(w) \cdot f(x, y, z, w) \cdot \equiv \cdot (\exists x)\sim(\exists y)\sim(\exists z)\sim(\exists w) \cdot \sim f(x, y, z, w).$$

And since $[x] \cdot \phi x \cdot \equiv \cdot \sim (\exists x) \cdot \phi x$, we have, as an equivalent of the first of these functions, $[x]\,[y]\,[z]\,[w] \cdot f(x, y, z, w)$, and, as an equivalent of the second,

$$\sim[x]\,[y]\,[z]\,[w] \cdot \sim f(x, y, z, w).$$

Let

$$(x, \cdots, l)(\exists y, \cdots, m) \cdots (\exists z, \cdots, n) \cdot f(x, \cdots, l;\ y, \cdots, m; \cdots ;\ z, \cdots, n)$$

be a function in reduced form. In this function the variables of widest scope are quantified universally, and the degree of quantification of the function is even.

$$\begin{aligned} &(x, \cdots, l)(\exists y, \cdots, m) \cdots (\exists z, \cdots, n) \\ &\qquad \cdot f(x, \cdots, l;\ y, \cdots, m; \cdots ;\ z, \cdots, n) \\ \cdot \equiv \cdot &\sim (\exists x, \cdots, l) \sim (\exists y, \cdots, m) \cdots \sim (\exists z, \cdots, n) \\ &\qquad \cdot f(x, \cdots, l;\ y, \cdots, m; \cdots ;\ z, \cdots, n) \\ \cdot \equiv \cdot &[x, \cdots, l][x, \cdots, m] \cdots [z, \cdots, n] \\ &\qquad \cdot f(x, \cdots, l;\ y, \cdots, m; \cdots ;\ z, \cdots, n). \end{aligned}$$

This last function is expressed in terms of the applicative [ ] alone, and it does not involve "$\sim$" in the quantifier. Every change of bracket has the force of a change of quantity from universal to particular or from particular to universal. This is so because the force of [ ] is negative. For example, the proposition "Every element has a successor," which may be written $(x) : (\exists y) \cdot x<y$, is equivalent to "No element is without a successor," which may be written $[x] : [y] \cdot x<y$, there is *no* x such that *no* y is such that $x<y$. Propositions whose variables of widest scope are quantified universally do not entail existence; that is to say, they would be true if there were not at least one element within the range of significance of the variables.

Let $(x, \cdots, l)\ (\exists y, \cdots, m) \cdots (z, \cdots n) \cdot f(x, \cdots, l;\ y, \cdots, m; \cdots ;\ z, \cdots, n)$

be a function in reduced form. Its degree of quantification is odd and its variables of widest scope are quantified universally.

$$
\begin{aligned}
&(x, \cdots, l)(\exists y, \cdots, m) \cdots (z, \cdots, n)\\
&\quad \cdot f(x, \cdots, l ; y, \cdots, m ; \cdots ; z, \cdots, n)\\
\cdot \equiv \cdot &\sim (\exists x, \cdots, l) \sim (\exists y, \cdots, m) \cdots \sim (\exists z, \cdots, n)\\
&\quad \cdot \sim f(x, \cdots, l ;y, \cdots, m ; \cdots ; z, \cdots, n)\\
\cdot \equiv \cdot &[x, \cdots, l][y, \cdots, m] \cdots [z, \cdots, n]\\
&\quad \cdot \sim f(x, \cdots, l;y, \cdots, m ; \cdots ; z, \cdots, n).
\end{aligned}
$$

Here "$\sim$" appears before the elementary function, but it is not involved in the quantifier. For example, in connection with serial relations, the proposition "Every element has an immediate successor" may be expressed by

$$(x)(\exists y)(z) : x < y : x < z \cdot y < z \cdot \vee \cdot z < x \cdot z < y,$$

which is equivalent to

$$[x][y][z] : \sim (x < y : x < z \cdot y < z \cdot \vee \cdot z < x \cdot z < y).$$

Let $(\exists x, \cdots, l) (y, \cdots, m) \cdots (z, \cdots, n)$
$\quad \cdot f(x, \cdots, l; y, \cdots, m; \cdots ; z, \cdots, n)$

express a function in reduced form. The degree of quantification is even and the variables of widest scope are quantified particularly. The function is equivalent to

$$
\begin{aligned}
&\sim \sim (\exists x, \cdots, l) \sim (\exists y, \cdots, m) \cdots \sim (\exists z, \cdots, n)\\
&\quad \cdot \sim f(x, \cdots, l ; y, \cdots, m ; \cdots ; z, \cdots, n),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\sim [x, \cdots, l][y, \cdots, m] \cdots [z, \cdots, n]\\
&\quad \cdot \sim f(x, \cdots, l; y, \cdots, m ; \cdots ; z, \cdots, n).
\end{aligned}
$$

Here "$\sim$" appears before the entire function. This is necessary since [ ] is universal and the function to be expressed is particular in respect of its variables of widest scope. Propositions of this form entail existence; they would be false if there were not at least one element within the range of significance of the variables.

$$\begin{aligned} &(\exists x, \cdots, l)(y, \cdots, m) \cdots (\exists z, \cdots, n) \\ &\quad \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n) \\ \cdot \equiv \cdot &\sim\ \sim(\exists x, \cdots, l) \sim(\exists y, \cdots, m) \cdots \sim(\exists z, \cdots, n) \\ &\quad \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n) \\ \cdot \equiv \cdot &\sim [x, \cdots, l][y, \cdots, m] \cdots [z, \cdots, n) \\ &\quad \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n). \end{aligned}$$

It is clear that if a function is to be expressed in terms of [ ] it must be in reduced form. Functions expressed in terms of "some" and "every" have a multiplicity of forms which do not appear in functions on [ ]. Thus

$$\begin{aligned} &(x)(y)(\exists z)(\exists w) \cdot f(x,y,z,w) \cdot \equiv \cdot (x,y)(\exists z)(\exists w) \\ &\quad \cdot f(x,y,z,w) \\ \cdot \equiv \cdot &(x)(y)(\exists z,w) \cdot f(x,y,z,w) \cdot \equiv \cdot (x,y)(\exists z,w) \\ &\quad \cdot f(x,y,z,w). \end{aligned}$$

Any one of these functions is equivalent to

$$[x,y][z,w] \cdot f(x,y,z,w),$$

and there is no other equivalent form in this quantifier; a change of bracket is always significant, and no two functions differing as to degree of quantification can be strictly equivalent.

Since $(\exists x) \cdot \phi x \cdot \equiv \cdot \sim (x) \cdot \sim \phi x$, any function in "some" and "every" can be expressed as a function in "every" and "$\sim$". Thus

$$\begin{aligned} &(x, \cdots, l)(\exists y, \cdots, m) \cdots (\exists z, \cdots, n) \\ &\quad \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n) \\ \cdot \equiv \cdot &\sim\ \sim(x, \cdots, l) \sim(y, \cdots, m) \cdots \sim(z, \cdots, n) \\ &\quad \cdot \sim f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n), \end{aligned}$$

and since

$$\{x, \cdots, n\} \cdot \phi(x, \cdots, n) \cdot \equiv \cdot \sim(x, \cdots, n) \cdot \phi(x, \cdots, n),$$

this function is equivalent to

$$\sim \{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\} \cdot \sim f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n).$$

Here we cannot dispense with "$\sim$" as affecting the entire function, since the force of the applicative $\{\ \ \}$, when it occurs in the first place, is particular, whereas the applicative in the first place of the function to be expressed is universal.

$$(x, \cdots, l)(\exists y, \cdots, m) \cdots (z, \cdots, n) \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \sim \sim (x, \cdots, l) \sim (y, \cdots, m) \cdots \sim (z, \cdots, n) \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \sim \{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\} \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n).$$

Also,

$$(\exists x, \cdots, l)(y, \cdots, m) \cdots (\exists z, \cdots, n) \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \sim (x, \cdots, l) \sim (y, \cdots, m) \cdots \sim (z, \cdots, n) \cdot \sim f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\} \cdot \sim f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n).$$

And

$$(\exists x, \cdots, l)(y, \cdots, m) \cdots (z, \cdots, n) \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \sim (x, \cdots, l) \sim (y, \cdots, m) \cdots \sim (z, \cdots, n) \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n)$$
$$\cdot \equiv \cdot \{x, \cdots, l\}\{y, \cdots, m\} \cdots \{z, \cdots, n\} \cdot f(x, \cdots, l; y, \cdots, m; \cdots; z, \cdots, n).$$

Heretofore we have been concerned with functions which involve a single complex quantifier, and in which, accordingly, any two variables have interdependent scopes, and each variable has within its scope the whole of the elementary function. There are two kinds of functions which do not have this form. (1) A function may be such that, although any two variables have interdependent scopes, at least one variable does not have the entire elementary function within its scope. The function

$$(y) : \phi y \cdot \vee \cdot (\exists x) \cdot \psi(x, y)$$

is such a function; or the function

$$(y) : \phi y \cdot \vee \cdot (\exists x) \cdot \psi x.$$

It is to be noted, with regard to this latter function, that since ψx is within the scope of (y), the variable y may occur in ψx. (2) A function may be such that at least two variables have independent scopes. Thus

$$(y) : (\exists x) \cdot \phi(x, y) \cdot \vee \cdot (z) \cdot \psi(y, z)$$

and

$$(y) \cdot \phi y \cdot \vee \cdot (\exists z) \cdot \psi z$$

are such functions.

It is clear that any function can be so expressed that "$\sim$", when it occurs, does not have any quantifier within its scope. For $\sim(x) \cdot \phi x \cdot \equiv \cdot (\exists x) \cdot \sim \phi x$ and $\sim(\exists x) \cdot \phi x \cdot \equiv \cdot (x) \cdot \sim \phi x$, and $\sim(\phi x \cdot \psi y) \cdot \equiv \cdot \sim\phi x \vee \sim\psi y$ and $\sim(\phi x \vee \psi y) \cdot \equiv \cdot \sim \phi x \cdot \sim\psi y$. Now it can be shown that for any function there is a function in a single complex quantifier which is equivalent to it,* so that every function can be *expressed* in this form. We may consider functions of the first kind first. Let ϕx and ψy be functions such that ϕx does not involve the variable y. Then $(y) \cdot \phi x \vee \psi y$ entails $\phi x \cdot \vee \cdot (y) \cdot \psi y$. Conversely, if $\phi x \cdot \vee \cdot (y) \cdot \psi y$, then $(y) \cdot \phi x \vee \psi y$; so that $(y) \cdot \phi x \vee \psi y : \equiv : \phi x \cdot \vee \cdot (y) \cdot \psi y$. Similarly, $(\exists y) \cdot \phi x \vee \psi y : \equiv : \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$. Also $(y) \cdot \phi x \cdot \psi y : \equiv : \phi x : (y) \cdot \psi y$ and $(\exists y) \cdot \phi x \cdot \psi y : \equiv : \phi x : (\exists y) \cdot \psi y$. By a repeated application of these equivalences it is possible to transform any function of the kind (1) into a function which involves a single complex quantifier.

*See *Principia Mathematica*, vol. I, *9.

In dealing with functions of the kind (2), we may require the equivalences just given, but some one or more of the following relations are always necessary: $(x) \cdot \phi x : (y) \cdot \psi y : \equiv : (x) : (y) \cdot \phi x \cdot \psi y$; $(\exists x) \cdot \phi x : (\exists y) \cdot \psi y : \equiv : (\exists x) : (\exists y) \cdot \phi x \cdot \psi y$; $(x) \cdot \phi x \cdot \vee \cdot (y) \cdot \psi y : \equiv : (x) : (y) \cdot \phi x \vee \psi y$; $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y : \equiv : (\exists x) : (\exists y) \cdot \phi x \vee \psi y$. In each of these equivalences, as in those which follow, it must, of course, be true that ϕx does not involve the variable y and that ψy does not involve the variable x. $(x) \cdot \phi x : (\exists y) \cdot \psi y$ entails $(\exists x) : (y) \cdot \phi x \cdot \psi y$ and $(\exists x) : (y) \cdot \phi x \cdot \psi y$ entails $(x) \cdot \phi x : (\exists y) \cdot \psi y$, so that $(x) \cdot \phi x : (\exists y) \cdot \psi y : \equiv : (\exists y) : (x) \cdot \phi x \cdot \psi y$. But it is to be noted that although $(x) \cdot \phi x : (\exists y) \cdot \psi y$ entails $(x) : (\exists y) \cdot \phi x \cdot \psi y$, it is not the case that $(x) \cdot \phi x : (\exists y) \cdot \psi y$ follows from $(x) : (\exists y) \cdot \phi x \cdot \psi y$. This relation will be dealt with in detail presently since it involves a point about which mistakes have frequently been made. $(x) \cdot \phi x) \cdot \vee \cdot (\exists y) \cdot \psi y$ entails $(x) : (\exists y) \cdot \phi x \vee \psi y$ and $(x) : (\exists y) \cdot \phi x \vee \psi y$ entails $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$, so that these functions are equivalent. But, although $(\exists y) : (x) \cdot \phi x \vee \psi y$ entails $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$, $(\exists y) : (x) \cdot \phi x \vee \psi y$ does not follow from $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$. By a repeated application of the foregoing equivalences any function of the kind (2) can be expressed in a form which involves a single complex quantifier.

We have noted that $(\exists y) : (x) \cdot \phi x \vee \psi y$ does not follow from $(x) \cdot \phi x \cdot \vee \cdot (\exists y) \cdot \psi y$ and that $(x) \cdot \phi x : (\exists y) \cdot \psi y$ does not follow from $(x) : (\exists y) \cdot \phi x \cdot \psi y$. It has been pointed out that propositions of the form $(x) \cdot \phi x$ would be true if there were not at least one individual within the range of significance of the variables. For $(x) \cdot \phi x$ entails $\sim(\exists x) \cdot \sim \phi x$ and $\sim(\exists x) \cdot \sim\phi x$ entails $(x) \cdot \phi x$; $(\exists x) \cdot \sim\phi x$ entails $\sim(x) \cdot \phi x$ and $\sim(x) \cdot \phi x$ entails $(\exists x) \cdot \sim\phi x$. Accordingly, $(x) \cdot \phi x$ and $(\exists x) \cdot \sim\phi x$ are proper contradictories, — $(x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ is necessary and $(x) \cdot \phi x : (\exists x) \cdot \sim \phi x$ is impossible. Now the proposition $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ is not a necessary proposition.

It is empirically significant in that it entails the existence of at least one individual, and this is not a fact which is certifiable on formal grounds alone. If it is not the case that $(x) \cdot \phi x$ would be true if there were not at least one value for x, then $(x) \cdot \phi x$ entails $(\exists x) \cdot \phi x$. Accordingly, $(x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ entails $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$, and this latter proposition is necessary since it follows from a necessary proposition. But it is not the case that $(\exists x) \cdot \phi x \cdot \vee \cdot (\exists x) \cdot \sim \phi x$ is a necessary proposition.

It follows that the function $(x) : (\exists y) \cdot \phi x \vee \psi y$ does not make an existence demand. Any value of this function may be read: It is false that there is at least one value of $(\exists y) \cdot \phi x \vee \psi y$, say $(\exists y) \cdot \phi x_1 \vee \psi y$, such that $(\exists y) \cdot \phi x_1 \vee \psi y$ is false. This will be true if there are no values. On the other hand, any value of the function $(\exists y) : (x) \cdot \phi x \vee \psi y$ may be read: There is at least one value of $(x) \cdot \phi x \vee \psi y$, say $(x) \cdot \phi x \vee \psi y_1$, such that $(x) \cdot \phi x \vee \psi y_1$ is true. This requires at least one value for y. Accordingly, these two functions are not strictly equivalent.

We have shown that any function can be so expressed as to have a single complex quantifier. It follows that every function has either the form

$$(x, \cdots, l)(\exists y, \cdots, m)(z, \cdots, n) \cdots f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n; \cdots)$$

or the form

$$(\exists x, \cdots, l)(y, \cdots, m)(\exists z, \cdots, n) \cdots f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n; \cdots).$$

The first of these functions can be expressed in terms of [ ] alone and the second can be expressed in terms of { } alone. Consequently, any function can be so expressed as to have one of the forms

$$[x, \cdots, l][y, \cdots, m][z, \cdots, n] \cdots f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n; \cdots)$$

or

$$\{x, \cdots, l\}\{y, \cdots, m\}\{z, \cdots, n\} \cdots f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n; \cdots).$$

In accordance with these forms, any function of whatever degree of quantification can be given in terms of a single applicative. One of every pair of mutually contradictory propositions can be given in terms of [ ] alone and the other can be given in terms of { } alone. $[x, \cdots, n] \cdot f(x, \cdots, n)$ is equivalent to $\sim\{x, \cdots, n\} \cdot \sim f(x, \cdots, n)$ and $\{x, \cdots, n\} \cdot f(x, \cdots, n)$ is equivalent to $\sim [x, \cdots, n] \cdot \sim f(x, \cdots, n)$. Similarly, $[x, \cdots, l]$ $[y, \cdots, m]$ $[z, \cdots, n]$ $\cdots f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$ is equivalent to $\sim \{x, \cdots, l\}$ $\{y, \cdots, m\}$ $\{z, \cdots, n\} \cdots \sim f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$ and $\{x, \cdots, l\}$ $\{y, \cdots, m\}$ $\{z, \cdots, n\}$ $\cdots f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$ is equivalent to $\sim[x, \cdots, l]$ $[y, \cdots, m]$ $[z, \cdots, n]$ $\cdots$ $\sim f(x, \cdots, l; y, \cdots, m; z, \cdots, n; \cdots)$.

The function $(x) \cdot \phi x$ is a generalization of the conjunctive function $\phi x_1 \cdot \phi x_2 \cdot \phi x_3 \cdots \phi x_n$, which involves a finite number of the values of ϕx. Similarly, $(\exists x) \cdot \phi x$ is a generalization of the finite disjunctive function

$$\phi x_1 \vee \phi x_2 \vee \phi x_3 \vee \cdots \vee \phi x_n.$$

In precisely the same way $[x] \cdot \phi x$ is a generalization of

$$\sim\phi x_1 \cdot \sim\phi x_2 \cdot \sim\phi x_3 \cdots \sim\phi x_n.$$

Now this latter function has the same force as Sheffer's "stroke" function $p|q$,* when $p|q$ is rendered $\sim p \cdot \sim q$. Accordingly, $[x] \cdot \phi x$ is a generalization of $\phi x_1|\phi x_2$, when interpreted ϕx_1 is false and ϕx_2 is false. Similarly, $\{x\} \cdot \phi x$ is a generalization of

$$\sim\phi x_1 \vee \sim\phi x_2 \vee \sim\phi x_3 \vee \cdots \vee \sim\phi x_n.$$

This function has the same force as the "stroke" function $p|q$ in its alternative interpretation $\sim p \vee \sim q$; so that $\{x\} \cdot \phi x$ generalizes $\phi x_1|\phi x_2$ in this interpretation.

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* See H. M. Sheffer, *A set of five independent postulates for boolean algebras*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 481–488.