AN INVARIANT OF A GENERAL TRANSFORMATION OF SURFACES*

BY W. C. GRAUSTEIN

1. *Introduction.* If two surfaces, S and S', are in one-to-one point correspondence, the transformation T of S into S' establishes between the pencils of tangent lines at corresponding points of S and S' a projective correspondence. Furthermore, if the line of intersection, L, of the tangent planes to S and S' at the corresponding points M and M' passes through neither of these points, that is, if neither S nor S' is a focal surface of the congruence of lines MM', the pencils of tangent lines at M and M' cut L in projective ranges of points.

The invariant cross ratio of the projectivity on L is an invariant of the transformation T with respect to the group of collineations of the three-dimensional space in which S and S' are imbedded. We propose to study this invariant, and to apply it, in particular, to the so-called fundamental transformations of surfaces.

2. *General Case.* We shall restrict ourselves primarily to the general case in which the projective correspondence on L has two distinct fixed points, D_1 and D_2. Let the surfaces S and S' be represented parametrically so that corresponding points have the same curvilinear coordinates (u,v). In particular, take as the u-curves the corresponding families of curves on S and S' whose tangents at corresponding points, M and M', intersect in D_1 and, as the v-curves, the curves whose tangents at corresponding points intersect in D_2.

A. *Fixed Points Finite.* If D_1 and D_2 are both finite points expressions for their coordinates, $y^{(1)} : (y_1^{(1)}, y_2^{(1)}, y_3^{(1)})$ and $y^{(2)} : (y_1^{(2)}, y_2^{(2)}, y_3^{(2)})$ are readily found. Since, for example,

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D_1 lies on the tangent at M to the u-curve on the surface S which passes through M, there exists a scalar function $\lambda(u,v)$, such that $y^{(1)}=x+\lambda x_u$, where $x=x(u,v): x_1(u,v)$, $x_2=x_2(u,v)$, $x_3=x_3(u,v)$ is the parametric representation of the surface S. Proceeding after this fashion, we obtain the equations

$$(1a)\qquad \begin{cases} y^{(1)}=x+\lambda x_u\,, \\ y^{(2)}=x+\mu x_v\,, \end{cases} \qquad \begin{aligned} y^{(1)}&=x'+\lambda' x'_u\,, \\ y^{(2)}&=x'+\mu' x'_v\,. \end{aligned}$$

Let the equations $u=u(t)$, $v=v(t)$ represent corresponding curves lying on S and S' and passing respectively through M and M', and denote by P and P' the points in which the tangents at M and M' to these curves intersect L. The coordinates y and y' of P and P' can evidently be written in the forms

$$y=x+\rho\frac{dx}{dt}\,, \qquad y'=x'+\rho'\frac{dx'}{dt}\,,$$

where $\rho(u,v)$ and $\rho'(u,v)$ are scalar functions.

Since P is collinear with D_1 and D_2,

$$y=a_1y^{(1)}+a_2y^{(2)}, \qquad a_1+a_2\equiv 1.$$

The values of a_1 and a_2 are readily found to be

$$a_1=\frac{\rho}{\lambda}u'(t)\,, \qquad a_2=\frac{\rho}{\mu}v'(t).$$

Consequently, the ratio in which the point P divides the line segment D_1D_2 is

$$\frac{a_2}{a_1}=\frac{\lambda}{\mu}\,\frac{v'(t)}{u'(t)}\,.$$

Similarly, the ratio in which P' divides D_1D_2 is

$$\frac{a'_2}{a'_1}=\frac{\lambda'}{\mu'}\,\frac{v'(t)}{u'(t)}.$$

Hence, the invariant cross ratio of the projectivity on L, namely, the cross ratio I:

$$I=(D_1D_2PP')\,,$$

has the value

$$I = \frac{\lambda}{\mu} \Big/ \frac{\lambda'}{\mu'} = \frac{\lambda\mu'}{\lambda'\mu}\,.$$

B. *Fixed Points at Infinity.* If D_1 and D_2 are both points at infinity, L is a line at infinity and the tangent planes to S and S' at corresponding points are parallel : S and S' correspond by a parallel map.*

Since corresponding parametric curves have at corresponding points parallel tangents, scalar functions, λ, λ', μ, μ', exist so that

$$(1b) \qquad \begin{cases} \lambda x_u = \lambda' x'_u\,, \\ \mu x_v = \mu' x'_v\,. \end{cases}$$

By means of these equations it is readily shown that the invariant I of the transformation of S into S' is given by the same formula as in the preceding case.

C. *One Fixed Point at Infinity, the Other Finite.* In this case corresponding tangent planes are not, in general, parallel but there exist two families of curves, one on each surface, which correspond and have at corresponding points parallel tangents. A map of this type we shall call *semi-parallel.*

If D_1 is the fixed point at infinity, it is the u-curves on S and S' whose corresponding tangents are parallel. Hence, we can write

$$(1c) \qquad \begin{gathered} \lambda x_u = \lambda' x'_u\,, \\ y^{(2)} = x + \mu x_v\,, \qquad y^{(2)} = x' + \mu' x'_v\,. \end{gathered}$$

Here, too, we find that I is given by the same formula as before.

In all three cases in which the projectivity on L has two distinct fixed points, the invariant I is given by the formula

$$(2) \qquad I = \frac{\lambda\mu'}{\lambda'\mu}\,.$$

* For maps of this type the invariant I was introduced and extensively employed in a previous paper by the author: *Parallel maps of surfaces*, TRANSACTIONS OF THIS SOCIETY, vol. 23 (1922), pp. 298–332.

3. *Associated Congruence.* Before leaving the case of two fixed points, it is worth while to note certain relationships between the transformation of S into S' and the congruence of lines joining corresponding points of S and S'. It is geometrically evident that the focal planes of the congruence which pass through the line MM' are the planes determined by MM' and the fixed points, D_1 and D_2, of the correspondence on L. Hence, the developables of the congruence are the surfaces $v=\text{const.}$ and $u=\text{const.}$

Let F_1 and F_2 be the focal points on MM' which are associated respectively with the developables $v=\text{const.}$ and $u=\text{const.}$ The cross ratio, $(F_2F_1\ MM')$, of the four points on MM' is equal, by a theorem of Chasles,* to the cross ratio of the tangent planes at these points to the ruled surface $u=u(t)$, $v=v(t)$ of the congruence. But these planes are precisely the planes through MM' determined respectively by the four points D_1, D_2, P, P' on L. Consequently,

$$I=(D_1D_2PP')=(F_2F_1MM') . \tag{3}$$

The value of the invariant of the transformation of S into S' for an arbitrary pair of corresponding points, M and M', is equal to a cross ratio in which the points M, M' are divided by the focal points of the associated congruence which lie on their line.

Since, by equations (2) and (3),

$$(F_2F_1MM')=\frac{\lambda\mu'}{\lambda'\mu} ,$$

it is reasonable to predict the following theorem.

The ratios in which the focal points of the congruence which lie on the line MM' divide the line-segment MM' are λ'/λ and μ'/μ.

In fact, it is a simple matter to verify the fact that the coordinates, $x^{(1)}$ and $x^{(2)}$, of the focal points F_1 and F_2 are, in all three cases of § 2,

$$x^{(1)}=\frac{\lambda'x'-\lambda x}{\lambda'-\lambda} , \qquad x^{(2)}=\frac{\mu'x'-\mu x}{\mu'-\mu} .$$

* Journal de Mathématiques, (1), vol. 2 (1837), p. 413.

4. *Radial Transformations.* If the projective correspondence on L is always the identity, every ruled surface of the congruence of lines MM' is a developable, and conversely. But a characteristic of a congruence all of whose ruled surfaces are developables is that its lines all pass through a point, finite or at infinity.

A necessary and sufficient condition that the projective correspondence on L *be always the identity is that the lines of the congruence joining corresponding points of* S *and* S' *all go through a point.*

A transformation of S into S' of this type we shall call a *radial* transformation. The invariant I shall have the value unity.

5. *Fundamental Transformations.* If the developables of the congruence of lines MM' cut the surfaces S and S' in conjugate nets, that is, if the parametric curves on both S and S' form conjugate nets, the transformation of S into S' is called a fundamental transformation or a transformation F.* For example, a parallel transformation is a transformation F. A semi-parallel transformation is not, in general.

Eisenhart's method† of establishing the general transformation F of a surface S into a second surface S', or, better, of a conjugate net N into a second (conjugate) net N' is equivalent to expressing F as the product of a parallel, a radial, and a second parallel transformation :

$$F: \qquad N \xrightarrow{P_1} \bar{N} \xrightarrow{R_2} \bar{N}' \xrightarrow{P_v} N',$$

where the intermediate transforms, $\bar{N}$ and $\bar{N}'$, of N are themselves nets.

The transformation P_1 is an arbitrary parallel transformation

$$P_1: \qquad \bar{x}_u = hx_u, \qquad \bar{x}_v = lx_v.$$

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* Cf. Jonas, SITZUNGSBERICHTE BERLINER MATHEMATISCHE GESELLSCHAFT, vol. 14 (1915), pp. 96–118; Eisenhart, TRANSACTIONS OF THIS SOCIETY, vol. 18 (1917), pp. 97–124, *Transformations of Surfaces*, Princeton University Press, 1923.

† *Transformations of Surfaces*, pp. 34–36.

If θ is a solution of the point equation of the given net N and $\bar{\theta}$ is the "corresponding" solution of the point equation of the parallel net $\overline{N}$:

$$\bar{\theta}_u = h\theta_u \ , \quad \bar{\theta}_v = l\theta_v \ ,$$

the radial transformation R_2 is given by the equation

$$R_2 : \qquad \bar{x}' = \frac{\bar{x}}{\bar{\theta}} \ .$$

The second parallel transformation is, then,

$$P_v : \qquad x_u' = -\frac{\tau}{h}\bar{x}_u' \ , \qquad x_v' = -\frac{\sigma}{l}\bar{x}_v' \ ,$$

where

$$\sigma = \theta l - \bar{\theta} \ , \qquad \tau = \theta h - \bar{\theta} \ .$$

The invariants I_1, I_2, I_3 of the three transformations are, respectively,

$$I_1 = \frac{h}{l} \ , \qquad I_2 = 1 \ , \qquad I_3 = \frac{\tau}{\sigma}\,\frac{l}{h} .$$

Hence

$$I_1 I_2 I_3 = \frac{\tau}{\sigma} \ .$$

On the other hand, the transformation F, expressed directly, has the form

$$F : \qquad x' = x - \frac{\theta}{\bar{\theta}}\,\bar{x} \ ,$$

and the focal points F_1, F_2 of the congruence of lines MM' have the coordinates

$$x^{(1)} = x - \frac{1}{h}\,\bar{x} \ , \qquad x^{(1)} = x - \frac{1}{l}\,\bar{x} \ .$$

Thus, the invariant I of the transformation F is found, by application of the first theorem of § 3, to have the value

$$(4) \qquad I = (F_2 F_1 M M') = \frac{\tau}{\sigma} \ .$$

When the general transformation F of a net N is expressed in the Eisenhart form as the product of a parallel, a radial, and a second parallel transformation, the invariant of F is equal to the product of the invariants of the individual transformations.

Jonas also expresses the transformation F as the product of three transformations *

$$F: \qquad N \xrightarrow{R_1} \bar{N} \xrightarrow{P_2} \bar{N}' \xrightarrow{R_3} N' ,$$

where

$$R_1: \qquad \bar{x} = \frac{x}{\theta} ,$$

$$P_2: \qquad \bar{x}'_u = \tau \bar{x}_u , \qquad \bar{x}'_v = \sigma \bar{x}_v ,$$

$$R_3: \qquad x' = -\frac{\theta}{\bar{\theta}} \bar{x}' .$$

Since

$$I_1 = 1 , \qquad I_2 = \frac{\tau}{\sigma} , \qquad I_3 = 1 ,$$

we have in this case, also,

$$I = I_1 I_2 I_3 .$$

The invariant of the general transformation F of a net is equal to the product of the invariants of the radial, parallel, and second radial transformations into which it can be factored.

The theorem amounts to saying that the invariant of F is equal to that of the parallel transformation P_2.

6. *Theorem of Permutability.* Outstanding in the theory of transformations F is the theorem of permutability. According to this theorem, if a net N is given and the nets N' and N'' are two F transforms of N chosen at random, there exist ∞^2 nets $\bar{N}$, each of which is an F transform of both N' and N''.

* Cf. Eisenhart, *Transformations of Surfaces*, pp. 40–46.

The transformation T of the net N into a chosen one of the nets $\bar{N}$ can be effected in either of two ways. For, since

$$N \xrightarrow{F_1'} N' \xrightarrow{F_2'} \bar{N}\,, \qquad N \xrightarrow{F_1''} N'' \xrightarrow{F_2''} \bar{N}\,,$$

we have

$$T = F_1' F_2' \quad \text{or} \quad T = F_1'' F_2''.$$

The invariants of the fundamental transformations F_1', F_2', F_1'', F_2'' are, by (4),

$$I_1' = \frac{\tau_1'}{\sigma_1'}\,, \qquad I_2' = \frac{\tau_2'}{\sigma_2'}\,, \qquad I_1'' = \frac{\tau_1''}{\sigma_1''}\,, \qquad I_2''' = \frac{\tau_2''}{\sigma_2''}\,.$$

But*

$$\tau_1'\tau_2' = \tau_1''\tau_2''\,, \qquad \sigma_1'\sigma_2' = \sigma_1''\sigma_2''\,.$$

Consequently,

$$I_1' I_2' = I_1'' I_2''\,.$$

If a transformation of a net N into a net $\bar{N}$ is equivalent to each of two pairs of transformations F, as described in the theorem of permutability, the product of the invariants of the transformations of the one pair is equal to the product of the invariants of those of the other pair.

HARVARD UNIVERSITY

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* Cf. Eisenhart, *Transformations of Surfaces*, p. 50.