

SOME PHASES OF DESCRIPTIVE GEOMETRY*

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The purpose of this paper is to recall those phases of descriptive geometry which are involved in the construction of adequate pictures of space objects. These remarks are particularly addressed those who attempt to use pictures and thus admit a need for them. To quote from Klein, "Is it not as worthy an object of mathematics to be able to draw correctly as to be able correctly to calculate?"

The need for descriptive geometry was probably first felt by the artist and the architect. The former was interested in conveying by means of his drawings a clear notion of the spatial form of the object represented. The latter, on the other hand, made use of his process of drawing, not only to instruct the builder as to the form and size of the objects which he represented by his pictures, but, in addition, to solve by means of plane constructions the problems of space which were encountered by the mason and carpenter. Thus the art of stereotomy was developed in the Middle Ages. In separating the geometric constructions of this art from their application, Frézier, in 1738, laid the cornerstone of modern descriptive geometry. It was Gaspard Monge (1746-1818) however, who developed this new constructive geometry of space so as to elevate it to the dignity of a pure science to which he gave the name descriptive geometry.

At this point let us observe that the process of drawing can be performed only upon a surface, and, in particular, upon a plane surface. It thus becomes possible to execute graphically, i. e., with pencil, ruler and compasses (instruments of the geometer), the following fundamental operations of plane geometry (postulates of construction):

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- I 1. To determine the (straight) line connecting two points.
 2. To find the point of intersection of two lines.
 3. To construct a circle of given center and radius.

The corresponding fundamental operations of space are:

- II 1. To find the line connecting two points of space.
 2. To find the line of intersection of two planes.
 3. To find the plane determined by a point and a line.
 4. To find the point in which a line pierces a plane.
 5. To construct a sphere of given center and radius.

There is, however, no process in space which is analogous to that of drawing in the plane. One might thus be tempted to say that the problems of space are incapable of graphical solution. In the art of stereotomy, solutions of problems of type II were virtually reduced to those of problems of type I, thus opening the way for the graphical solution of space problems.*

The principal purposes of descriptive geometry thus appear to be the following:

1. Representation of the objects of space by means of figures which lie in a plane (or upon a surface).
2. Solution of the problems of space by means of constructions which can be executed in the plane.

The criterion of what shall be meant by a good plane representative of a space object has, more or less unconsciously, been taken to be *that the plane representative when properly placed, shall produce upon the retinal surface of the eye an image which differs but little from that produced by the object itself*. Since the optical properties of the eye may be compared with those of the camera obscura, it is easy to see† that *the above criterion may be satisfied, at least approximately, by a plane picture which*

* See Gino Loria, *Vorlesungen über Darstellende Geometrie*, Leipzig, Teubner, 1907.

† See A. Schoenflies, *Einführung in die Hauptgesetze der Zeichnerischen Darstellungsmethoden*, Leipzig, Teubner, 1908.

is a *central* (or, by accommodation, also a *parallel*) *projection*. A photograph is such a plane picture.

In order that the solution of a space problem may be made possible by a construction in the plane, it is essential that there exist an *unambiguous correspondence between space and the plane*. To make this clear, we will observe that a single projection does not satisfy this requirement. For, while to each point of space P there corresponds a definite projection P' on the picture plane, it is *not* true, conversely, that to each point of the projection there is a definite point of space. Hence, in general, a single projection does not suffice if no information concerning the object projected is known. The reason that a single projection, like a photograph, may convey an adequate notion of an object (like a building) is because we have some information about the object, such, for instance, as the perpendicularity of some of its edges. If, however, such a relation concerning the object to be represented is not known, or not true, one projection of it is not sufficient to convey to the mind an adequate notion of its form. The projection of a cube might, for instance, also be the projection of any hexahedron of which the vertices are on the lines connecting the vertices of the cube with the center of projection. That two projections do, in general, suffice to set up an unambiguous correspondence between space and the plane, is illustrated by the stereoscope, which enables the user to combine the images of two pictures taken from different points of view and thus obtain the impression of solidity or relief.

Let us now illustrate several of the principal methods of descriptive geometry, showing how, in each of these methods, use is made either of *two projections* or of *one projection and information about the object projected*. For this purpose let us use, as an object to be represented, a solid of revolution obtained by revolving around one of its sides $O''Z''$ the part $O''D''E''F''Z''$ of a rectangle which remains after a corner has been removed by a circular cut $E''F''$ (see Fig. II). The solid of revolution

thus obtained is bounded by two planes (generated by $O'' D''$ and $Z'' F''$), by a circular cylinder (generated by $D'' E''$) and by a torus (generated by the circle $E'' F''$). Let us further suppose that points have been taken on the circle $E'' F''$ in such a way that their (orthographic) projections on the line $O'' Z''$ are equally spaced. In the revolution, these points generate circles of latitude of which the planes are perpendicular to the axis $O'' Z''$ and are equally spaced.

Let us now choose several planes on which to project our solid. As a first plane of projection we will take the plane base π_1 (generated by $O'' D''$) and as a second plane of projection, a plane π_2 parallel to the axis of revolution $O'' Z''$. We will think of π_2 as coincident with the plane of our drawing, and of π_1 as intersecting π_2 in the ground line g_{12} . Then we will think of π_1 as rotated around g_{12} until it becomes coincident with π_2 . The plane of the paper (π_2) is then a Mongean drawing plane. In it Fig. I is the orthographic projection of our solid on π_1 and Fig. II is that on π_2 . Together these figures constitute a *Mongean representation* of our solid. It is easily seen that one of these projections alone is not sufficient, but that the two are sufficient. Thus we see that associated with a point of space P there are two points P', P'' , in the drawing plane which lie on the same perpendicular to the ground line, and, conversely, to such a pair of points in the drawing plane there corresponds a point in space (namely, the point with which was associated this pair). From this it follows that associated with a line of space p there is a pair of lines p', p'' of the drawing plane; and conversely, in general, to a pair of lines in the drawing plane, there corresponds a line in space (namely, the line with which was associated this pair). A line of space p instead of being represented by its projections (p', p'') may also be represented by the points (P_1, P_2) (traces) in which it pierces the planes of projection π_1 , and π_2 (see drawing). It is the extension of the latter method which is used to represent a plane. Thus a plane μ is represented by the lines (m_1, m_2), (traces)

Fig. III

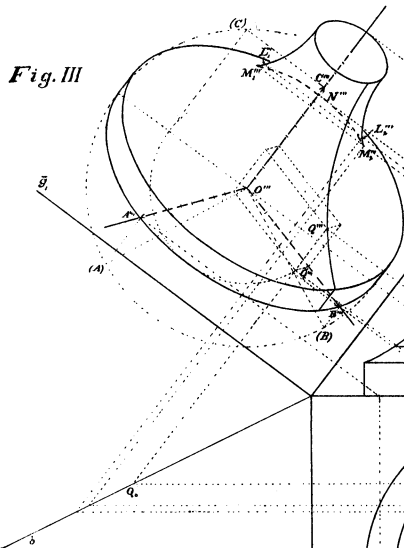


Fig. II

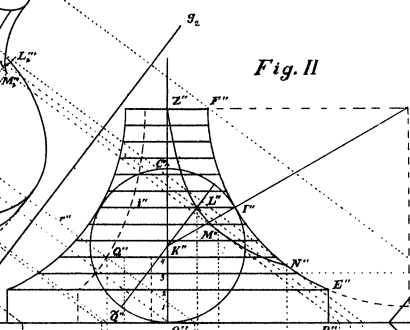


Fig. I

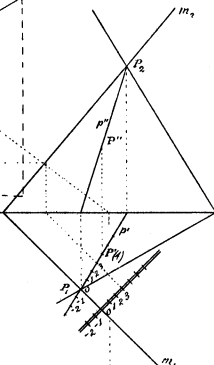
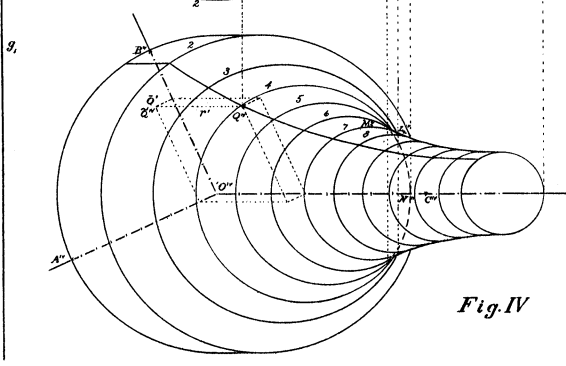


Fig. IV



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in which it cuts the planes π_1 and π_2 (see drawing). Since these traces necessarily intersect on the ground line g_{12} , it follows that a general plane is adequately represented by two lines of the drawing plane which intersect on the ground line. It is thus evident that the Mongean method of representing space (by means of its elements, points, lines, planes) in the (drawing) plane, possesses the property of unambiguous correspondence.

A circle of latitude, such as that generated by the point I'' , is represented by the circle i' in Fig. I and the straight line i'' in Fig. II. The corresponding representatives of other circles of latitude are also shown in these figures. These circles might be regarded as contour (level) lines if we now think of our solid as a mountain peak. If we attach to the circles in Fig. I numbers ($\dots, 2, 3, 4, \dots$) representing the distances of these contour lines above the (datum) plane π_1 , Fig. I becomes a *topographic representation* of our solid. In this method, a point is adequately represented by its (orthographic) projection and a number indicating its altitude. A (straight) line is represented by its projection and a scale, the intervals of which are the projections of the intervals into which the line in space is divided by equally spaced level planes; and finally, a plane is represented by its line of greatest slope. Thus, in the drawing, the point P' with the accompanying number (4) represents the point P (represented in the Mongean method by the pair of points P', P''), the line p' with the accompanying scale ($\dots, -2, -1, 0, 1, 2, \dots$) represents the line p (represented in the Mongean method by the pair of lines p', p''), and the double line m' with the accompanying scale ($\dots, -2, -1, 0, 1, 2, \dots$) represents the plane μ (represented in the Mongean method by the traces m_1, m_2). This method of representation also possesses the property of unambiguous correspondence.

Let us now find the (orthographic) projection (Fig. III) of our solid on a plane $\pi \equiv [g_1, g_2]$ which is inclined to π_1 , but perpendicular to π_2 . The horizontal trace g_1 of

π will be perpendicular to g_{12} . We will then think of the plane π as rotated around its vertical trace g_2 (just as π_1 was rotated around g_{12}) until it comes into coincidence with the plane of the drawing π_2 . By this rotation the trace g_1 assumes the new position \bar{g}_1 , perpendicular to g_2 at the intersection of g_2 with g_{12} . The two figures, Fig. II and Fig. III, together constitute another Mongean representation of our solid, with g_2 as the corresponding ground line.

The points of Fig. III may be obtained from those of Figs. I and II by a knowledge of the following facts:

If Q denote a general point of space and Q' , Q'' , Q''' its projections on π_1 , π_2 , π , respectively, then just as Q' and Q'' lie on the same perpendicular to the ground line g_{12} so Q'' and Q''' lie on the same perpendicular to the ground line g_2 . Furthermore, since the points Q' and Q''' are at equal distances from g_{12} and g_2 respectively, a line through Q' parallel to g_{12} meets a line through Q''' parallel to g_2 on the bisector b of the angle formed by g_1 and \bar{g}_1 (see the drawing). Thus it is easy to obtain point by point the projection on π of the circular edges of our solid (generated by D'' , E'' and F'').

In order to obtain the other bounding curve of Fig. III, let us observe that if we regard the rays which project our solid on the plane π as rays of light, this bounding curve is the projection on π of the *line of shade*, i. e., the curve which separates the illuminated portion of our solid from the unilluminated portion. Points of such a line of shade may be determined for a surface of revolution by the following well known construction. A surface of revolution may be regarded as the envelope of a one-parameter family of spheres of which the centers lie on the axis of revolution. Each sphere of this family has for its line of shade a great circle whose plane is perpendicular to the direction of the illuminating rays. On the other hand, each such sphere has contact with the envelope along a common circle of latitude of these two surfaces. The points in which these two circles intersect are points of the line of

shade of the envelope. To find the center of the sphere having contact with our surface along the circle of latitude generated by the point I'' , we erect at I'' in the drawing plane a normal to the meridian curve $E'' I'' F''$. This cuts the axis $O'' Z''$ in the required center K'' . Through K'' we then draw a line parallel to g_2 , since the rays of light are perpendicular to the plane π . This is the projection on π_2 of the line of shade of the sphere of center K , and cuts the corresponding projection i'' of the circle of contact in the point L'' . This point is the vertical projection of two points of our line of shade. The horizontal projections of these points are found by drawing through L'' a line perpendicular to g_{12} cutting the circle i' in the points L'_1 and L'_2 . The corresponding projections L''_1, L''_2 on the plane π (Fig. III) are then found by the construction given in the preceding paragraph. Thus the bounding curves of Fig. III corresponding to the toroidal surface of our solid, may be found point by point. Apparently these curves stop abruptly at the points M''_1 and M''_2 . However, these points are not "points d'arrêt", but they are *cusps* of the complete curve of which the remaining portion $M''_1 N''' M''_2$ is the bounding curve of the hidden (back) side of the torus. In fact this curve is a *parallel to the ellipse* into which the axial circle of the torus is projected on the plane π . For, the bounding curves of the (orthographic) projection of any tubular surface (i. e., envelope of a one-parameter family of spheres of constant radius) are parallels to the corresponding projection of the axial curve (i. e., locus of centers of enveloped spheres) of this surface, and the torus is a tubular surface with a circular axial curve.

With the same system of parallel projecting rays which were used to obtain the orthographic projection (Fig. III) of our solid on the plane π , let us also obtain the oblique projection (Fig. IV) of this solid on the plane π_1 . In order that Fig. IV may not be superimposed on Fig. I we will drop the former down a little distance on the paper. Hence, now instead of using Q' as the horizontal projection of Q ,

we will take \bar{Q}' as this projection. Then, to obtain the required oblique projection Q^{IV} of the point Q we merely find the points in which the ray r (r' , r'') through the point Q (\bar{Q}' , Q'') pierces the plane π .

Since the edges, and other circles of latitude, of our solid are parallel to the plane of (oblique) projection, each of these circles projects into an equal circle and all that is needed is merely to find the (oblique) projections of the centers of such circles. The envelope of the circles thus obtained is the bounding curve of the (oblique) projection (Fig. IV) of our solid. But this may be otherwise obtained by merely finding (as for the point Q) projections of points L of the line of shade.

If in Fig. IV (as in Fig. I) we attach numbers to the circles indicating the altitudes of the planes of the circular sections of which these circles are the (oblique) projection, Fig. IV alone is an adequate plane representation of our solid. This representation has the advantage over that given by Fig. I with the corresponding attached numbers, in that it conveys with less effort, on the part of an observer, a notion of the spatial form represented.

In a similar manner, Fig. III with numbered ellipses (not drawn) into which the circles of latitude of our solid project (orthographically), serves as an adequate plane representation of our solid.

However, instead of drawing these ellipses in Fig. III, or the circles in Fig. IV, we might merely give the scale ($O''' Z'''$ in Fig. III, $O^{IV} Z^{IV}$ in Fig. IV) into which the vertical scale OZ (shown in true size in Fig. I) projects. In addition to this scaled axis ($O''' Z'''$ in Fig. III or $O^{IV} Z^{IV}$ in Fig. IV) let us now take two other scaled axes ($O''' X'''$, $O''' Y'''$ in Fig. III or $O^{IV} X^{IV}$, $O^{IV} Y^{IV}$ in Fig. IV), these being the projections of space axes OX and OY , each bearing the scale of OZ and perpendicular to each other and to OZ . In addition to the projection of the general point Q (Q''' in Fig. III or Q^{IV} in Fig. IV), let us find that of the orthographic projection \tilde{Q} of the point Q on the

plane π_1 (\tilde{Q}''' in Fig. III or \tilde{Q}^{IV} in Fig. IV). Then this pair of points (Q''' , \tilde{Q}''' in Fig. III or Q^{IV} , \tilde{Q}^{IV} in Fig. IV) is sufficient to represent adequately a point of space Q . For we can find from the three axonometric axes ($O''' X'''$, $O''' Y'''$, $O''' Z'''$ in Fig. III or $O^{IV} X^{IV}$, $O^{IV} Y^{IV}$, $O^{IV} Z^{IV}$ in Fig. IV) the coordinates of the point Q in space.

We thus have the *axonometric method* of representation (orthographic in Fig. III, oblique in Fig. IV), in which two points on a line parallel to one of the axonometric axes represent adequately a point of space. Likewise, in general, two lines of the picture plane represent a line of space, and, finally, a plane of space is adequately represented by two lines of the picture plane which intersect on one of the axonometric axes.

While Figs. III and IV were obtained by Mongean processes from the Mongean representation consisting of Figs. I and II, they may be obtained directly by the axonometric method. In the use of this method, one must be able to determine the scales when the directions of the axes are given. The relationship which is involved between these angles and scales is, in the case of orthographic axonometry, expressed by the following theorem.

THEOREM. *Any two conjugate diameters and the minor axis of an ellipse may be regarded, as far as directions are concerned, as the orthographic projection on the plane of the ellipse of three mutually perpendicular axes; if these axes be taken as cartesian axes and the plane of the ellipse as the picture plane, the ratios borne by the halves of the chosen conjugate diameters and the focal distance of the ellipse to the half major axis of the ellipse are the foreshortening ratios, i. e., the numbers by which the scales on the space axes must be multiplied in order to obtain those on the axonometric axes. This relationship is indicated in Fig. III, in which the ratios of $O''' A'''$, $O''' B'''$, $O''' C'''$ to $O'''(A)$ are the foreshortening ratios.*

In the case of oblique axonometry, however, one may choose at pleasure both the angles and the scales. The

truth of this assertion is contained in the following famous theorem, which is due to Pohlke.

POHLKE'S THEOREM. *Three straight-line segments of arbitrary lengths in a plane, drawn from a point and making arbitrary angles with each other, can be regarded as a parallel projection of three equal segments drawn from the origin on three rectangular coordinate axes; however, not more than one of the given segments or one of the given angles can vanish*.*

Pictures of the type of Fig. III and Fig. IV make more of an appeal than do those of the type of Figs. I and II. In the case of most of the objects of architecture and technology there are among the bounding surfaces certain mutually perpendicular planes. These planes, or parallel planes, are usually selected as the Mongean planes of representation. Thus certain plane faces appear merely as lines, and hence these projections do not convey as easily a notion of the spatial form of the objects as do projections on planes not parallel to any of the mutually perpendicular planes. Hence, here also, pictures of the type of Fig. III and Fig. IV, which can easily be made directly by the axonometric method, prove very desirable.

All of the methods described in this paper possess the property of unambiguous correspondence. In each is given the plane representative of the space elements: point, line, plane. The first four of the space operations II thus have their counterparts in the picture plane and hence all of the three-dimensional problems of pure geometry of position can be solved in the plane by each of these methods. The fifth of the operations II can, in like manner, be replaced by the third of operations I, and thus also is made possible a graphical solution of all perpendicularity and metrical problems of space by means of constructions in the plane.

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* For a collection of proofs of this theorem see Wendling, *Der Fundamentalsatz der Axonometrie*, Zürich, Speidel, 1912.