SINGULARITIES OF CURVES OF GIVEN ORDER *

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1. *Introduction.* The following problems concerning plane curves are suggested by the theory of surfaces and space curves.

I. To find the greatest number of cusps that may be added to the singularities of a plane curve of order n with κ_1 cusps and δ_1 nodes.

II. To find the greatest number of nodes that may be added to the singularities of a plane curve of order n with κ_1 cusps and δ_1 nodes.

III. To find the greatest number of cusps that may occur among the remaining singularities of a plane curve of order n and genus p with κ_1 cusps and δ_1 nodes.

IV. To find the least order n of a plane curve that can possess a given number κ_1 of cusps and δ_1 of nodes.

These problems are all closely related to the question of the existence of plane curves with assigned singularities, in which the principal problem consists of finding the maximum number of cusps of a curve of given order and genus. This problem has been solved by Lefschetz,† who also mentions the first two of the above problems, but suggests only graphical solutions for them. The existence of curves with cusps up to and including the maximum has been shown by both Lefschetz and Coolidge ‡ for $p \leqq p_0$, where p_0 is the genus associated with a given n on the assumption that the bitangents and inflections are both as near zero as possible.

As a notation for the number of singularities of any algebraic plane curve of order n, class m, and genus p, we shall say that

* Presented to the Society, April 28, 1923.

† S. Lefschetz, *On the existence of loci with given singularities*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 23–41.

‡ J. L. Coolidge, *On the existence of curves with assigned singularities*, this BULLETIN, vol. 28 (1922), pp. 451–455.

the curve has κ cusps, δ nodes, ι inflections, and τ bitangents. We shall have occasion to use the following formulas, the first three of which are Plückerian:

$$(1)\quad m = n(n-1) - 2\delta - 3\kappa,$$

$$(2)\quad \iota = 3n(n-2) - 6\delta - 8\kappa,$$

$$(3)\quad 2\tau = 2\delta + (m-n)(m+n-9),$$

$$(4)\quad F = \tfrac{1}{2}n(n+3) - \delta - 2\kappa - 8 = 3n + p - 9 - \kappa, \qquad p \geqq 2.$$

The last formula, due to Professor Lefschetz,* gives the number of independent absolute invariants necessary to determine a curve of order n and genus $p \geqq 2$ with δ nodes and κ cusps.

The conditions for the maximum number of cusps of a curve of given order and genus are obtained by Lefschetz for certain limits of n in terms of a given p. If we restate them for certain limits of p in terms of a given n, we find that for $p \leqq p_0$, where p_0 is defined by

$$[p_0]\dagger \leqq \tfrac{1}{2}(n + 2 - \sqrt{4n+13}),$$

κ is a maximum when ι is a minimum; for $p_0 < p \leqq p_1$, where

$$[p_1] \leqq \tfrac{1}{2}(2n - 1 + \sqrt{16n-23}),$$

κ is a maximum when τ is a minimum; and for

$$p_1 \leqq p \leqq \tfrac{1}{4}(n-4)(n-5) \qquad \text{and} \qquad n > 13,$$

κ is limited by the number of independent absolute invariants determining the curve.

In all four problems, let κ_1 and δ_1 represent, respectively, the number of cusps and nodes already possessed by the curve of order n and let κ_m represent the maximum number of cusps that a curve of order n with δ_1 nodes may have. The solutions will now be given.

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* S. Lefschetz, loc. cit., p. 29.

† In this paper, the symbol $[x]$ followed by $\leqq$ ($\geqq$) means the largest (one greater than the largest if not equal) integer contained in the expression on the right. The interpretation not in parentheses is the usual one.

2. *Problem* I. From formula (2) we obtain, by means of the condition $\iota \geqq 0$,

(5) $$[\kappa_m] \leqq \tfrac{3}{8}(n^2 - 2n - 2\delta_1).$$

Substituting the value of m from (1) in (3), letting $\tau \geqq 0$, and solving for κ, we obtain

(6) $$[\kappa_m] \geqq \tfrac{1}{6}[2n(n-1) - 4\delta_1 - 9 - \sqrt{4n(n-9) - 8\delta_1 + 81}].$$

If we let $F \geqq 0$ in (4), we find

(7) $$[\kappa_m] \leqq \tfrac{1}{4}[n(n+3) - 2(\delta_1 + 8)].$$

For a given n and δ_1, the one of these three formulas that gives the least value of κ_m determines the maximum number of cusps that curve can have.

It is now necessary to find the limits of δ_1 for which each of these formulas applies. If

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-4),$$

the curve may be rational and have all the rest of its double points cusps. The limit (5) holds for $0 \leqq p \leqq p_0$ and for these curves

$$[\delta] \leqq \tfrac{1}{2}(n-2)(n-4) - 4p.$$

Replacing δ by δ_1, and eliminating p from this formula and the upper limit for p, we obtain

$$\delta_1 \geqq \tfrac{1}{2}n\,(n-10) + 2\sqrt{4n+13}.$$

To find the lower limit of δ_1 for formula (6) when $n > 13$, eliminate p from the inequality for p_1 and the formula for δ, when $p \geqq p_1$,

$$\delta = \tfrac{1}{2}(n-4)(n-5) - 2p.$$

This gives

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-11) - \sqrt{16n-23}.$$

The complete solution may now be stated as follows:

If

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-4),$$
$$\kappa_m = \tfrac{1}{2}(n-1)(n-2) - \delta_1.$$

If

$$\tfrac{1}{2}(n-2)(n-4) \geqq \delta_1 \geqq \tfrac{1}{2}n(n-10) + 2\sqrt{4n+13},$$
$$[\kappa_m] \leqq \tfrac{3}{8}(n^2 - 2n - 2\delta_1).$$

If

$$\tfrac{1}{2}n(n-10)+2\sqrt{4n+13}>\delta_1\begin{cases}\geqq 0 & \text{for} \quad n\leqq 13,\\ \geqq \tfrac{1}{2}(n-2)(n-11) - \sqrt{16n-23} & \text{for} \quad n>13,\end{cases}$$

$$[\kappa_m]\geqq \tfrac{1}{6}[2n(n-1)-4\delta_1-9-\sqrt{4n(n-9)-8\delta_1+81}].$$

If

$$\tfrac{1}{2}(n-2)(n-11)-\sqrt{16n-23}\geqq\delta_1\geqq 0 \quad \text{for} \quad n>13,$$

$$[\kappa_m]\leqq\tfrac{1}{4}[n(n+3)-2(\delta_1+8)].$$

In any of these cases, the number of cusps that may be added is $\kappa_m-\kappa_1$, and there may be one additional node if κ_m is not diminished by substituting δ_1+1 for δ_1 in the limit for κ_m.

3. *Problem* II. Let δ_m denote the greatest number of nodes (including δ_1) that the curve of order n with κ_1 cusps can have. Solve the three formulas (5), (6), (7) for δ in terms of n and κ_1 and find the limits of κ_1 for which they hold in the same way that the limits for δ_1 were found in the preceding problem In this case there can be no additional cusps. The results are as follows: If $\kappa_1\leqq\tfrac{3}{2}(n-2)$, we have

$$\delta_m=\tfrac{1}{2}(n-1)(n-2)-\kappa_1.$$

If

$$\tfrac{3}{2}(n-2)\leqq\kappa_1\leqq\tfrac{3}{2}(2n-\sqrt{4n+13}),$$

$$[\delta_m]\leqq\tfrac{1}{6}[3n(n-2)-8\kappa_1].$$

If

$$\tfrac{3}{2}(2n-\sqrt{4n+13})<\kappa_1\begin{cases}\leqq\tfrac{1}{3}n(n-2) & \text{for} \quad n\leqq 13,\\ \leqq\tfrac{1}{2}(8n-19+\sqrt{16n-23}) & \text{for} \quad n>13,\end{cases}$$

$$[\delta_m]\leqq\tfrac{1}{2}(n^2-n-3\kappa_1-5-\sqrt{3\kappa_1-8n+25}).$$

If

$$\tfrac{1}{2}(8n-19+\sqrt{16n-23})\leqq\kappa_1\leqq\tfrac{1}{4}(n^2+3n-16) \quad \text{for} \quad n>13,$$

$$[\delta_m]\leqq\tfrac{1}{2}(n^2+3n-16-4\kappa_1).$$

In any case, the number of nodes that may be added is given by the difference $\delta_m-\delta_1$.

4. *Problem* III. In this problem p, κ_1, and δ_1 must satisfy the inequality

$$\tfrac{1}{2}(n-1)(n-2)-\kappa_1-\delta_1 \geqq p \geqq \tfrac{1}{2}(n-1)(n-2)-\kappa_1-\delta_m.$$

The problem is solved by noting that the maximum number of cusps depends only on n and p when

$$\delta_1 \leqq \tfrac{1}{2}(n-1)(n-2)-\kappa_m-p,$$

and when δ_1 equals or exceeds this limit, all the remaining double points may be cusps. The following formulas result: If

$$0 \leqq p \leqq \tfrac{1}{2}(n+2-\sqrt{4n+13}),$$

and

$$\delta_1 \leqq \tfrac{1}{2}(n-2)(n-4)-4p,$$
$$[\kappa_m] \leqq \tfrac{3}{2}(n+2p-2).$$

If

$$\tfrac{1}{2}(n+2-\sqrt{4n+13}) < p \leqq \tfrac{1}{2}(2n-1+\sqrt{16n-23})$$

and

$$\delta_1 \leqq \tfrac{1}{2}(n^2-7n+13-6p+\sqrt{24p-8n+25}),$$
$$[\kappa_m] \leqq 2(n+p)-\tfrac{1}{2}(11+\sqrt{24p-8n+25}).$$

If

$$\tfrac{1}{2}(2n-1+\sqrt{16n-23}) \leqq p \leqq \tfrac{1}{4}(n-4)(n-5)$$

and

$$\delta_1 \leqq \tfrac{1}{2}(n-4)(n-5)-2p,$$
$$\kappa_m = 3n+p-9.$$

If

$$p > \tfrac{1}{4}(n-4)(n-5),$$

or if δ_1 equals or exceeds any of the above limits,

$$\kappa_m = \tfrac{1}{2}(n-1)(n-2)-\delta_1-p.$$

In any case, the number of cusps that may be added is $\kappa_m-\kappa_1$, and the additional nodes, if any, that belong to the curve, are given by the formula

$$\delta = \tfrac{1}{2}(n-1)(n-2)-\kappa_m-\delta_1-p.$$

The solution of the problem, *to find the maximum number of nodes that may occur among the remaining singularities of a curve of order n and genus p with κ_1 cusps and δ_1 nodes,* is evidently always

$$\delta_m - \delta_1 = \tfrac{1}{2}(n-1)(n-2) - \kappa_1 - \delta_1 - p,$$

since p cannot be less than the genus of a curve with κ_1 cusps and the maximum number of nodes.

5. *Problem* IV. For a rational curve we have

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-4), \tag{8}$$

$$\kappa_1 = \tfrac{1}{2}(n-1)(n-2) - \delta_1. \tag{9}$$

Solving (9) for n, we have

$$n = \tfrac{1}{2}[3 + \sqrt{8(\kappa_1 + \delta_1) + 1}].$$

Substituting this value of n in (8), and solving for δ_1, we obtain

$$\delta_1 \geqq \tfrac{2}{9}\kappa_1(\kappa_1 - 3),$$

which gives the relation between δ_1 and κ_1 for which the above formula holds.

When $\delta_1 < 2\kappa_1(\kappa_1 - 3)/9$ the curve cannot be rational and we must consider three cases, $0 \leqq p \leqq p_0$, $p_0 < p \leqq p_1$ and $p \geqq p_1$. For $p \leqq p_0$ the maximum number of cusps occurs when the number of inflections is a minimum. Solving the inequality resulting from equation (2) for n, we find

$$[n] \geqq \tfrac{1}{3}[3 + \sqrt{3(3 + 6\delta_1 + 8\kappa_1)}].$$

Eliminating p and n from the inequality defining p_0 and the two inequalities

$$[\delta_1] \geqq \tfrac{1}{2}(n-2)(n-4) - 4p,$$
$$[\kappa_1] \leqq \tfrac{3}{2}(n + 2p - 2),$$

which give the nodes and the maximum number of cusps for $p \leqq p_0$, we obtain

$$\delta_1 \geqq \tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}].$$

If $p_0 < p \leqq p_1$, the maximum number of cusps occurs when the number of double tangents is a minimum. If we attempt to solve the resulting inequalities in κ_1, δ_1 and n for n, we are led to the quartic equation

$$n^4 - 2n^3 - [2(3\kappa_1 + 2\delta_1) + 9]n^2 + 2(3\kappa_1 + 2\delta_1 + 9)n + (3\kappa_1 + 2\delta_1)(3\kappa_1 + 2\delta_1 + 9) + 2\delta_1 = 0.$$

A general solution of this equation is too involved to be of use as a formula, but the value of n for given values of κ_1 and δ_1 can be found as the least positive real root of this quartic when that root is an integer, or the integer next larger when the least positive root is irrational.

If

$$\tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}] > \delta_1 \geqq 0,$$

and $\kappa_1 \leqq 50$ the foregoing formula applies. If $\kappa_1 > 50$, however, the lower limit for δ_1, for which the foregoing formula applies, is not zero, but we cannot find it in general, since it involves the solution of two quartic equations.

If $\kappa_1 > 50$, and δ_1 is less than this lower limit, $p > p_1$, and the least value of n is obtained by eliminating p from the two formulas

$$\kappa_1 = 3n + p - 9,$$
$$p = \tfrac{1}{2}(n - 1)(n - 2) - \delta_1 - \kappa_1.$$

Solving these for n, we have

$$[n] \geqq \tfrac{1}{2}(\sqrt{16\kappa_1 + 8\delta_1 + 73} - 3).$$

The inequality sign is introduced in the result because δ_1 may not have the value that will satisfy the equality.

In general, when it is desired to find the least order n of a curve that can possess a given number κ_1 of cusps and δ_1 of nodes, we may proceed as follows. If

$$\delta_1 \geqq \tfrac{2}{9}\kappa_1(\kappa_1 - 3),$$
$$[n] = \tfrac{1}{2}[3 + \sqrt{8(\kappa_1 + \delta_1) + 1}].$$

If

$$\tfrac{2}{9}\kappa_1(\kappa_1 - 3) > \delta_1 \geqq \tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}],$$
$$[n] \geqq \tfrac{1}{3}[3 + \sqrt{3(3 + 6\delta_1 + 8\kappa_1)}].$$

If

$$\tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}] > \delta_1 \geqq 0 \quad \text{and} \quad \kappa_1 \leqq 50,$$

n is given as the least positive real root, if that is an integer, or the integer next larger than the least positive irrational root of the quartic

$$n^4 - 2n^3 - [2(3\kappa_1 + 2\delta_1) + 9]n^2 + 2(3\kappa_1 + 2\delta_1 + 9)n + (3\kappa_1 + 2\delta_1)(3\kappa_1 + 2\delta_1 + 9) + 2\delta_1 = 0.$$

If $\kappa_1 > 50$, determine n by the inequality

$$[n] \geqq \tfrac{1}{2}(\sqrt{16\kappa_1 + 8\delta_1 + 73} - 3).$$

If $p \geqq p_1$ for this value of n and the given values of κ_1 and δ_1, this is the correct value for n; but if $p < p_1$, this formula does not apply, and n must be determined from the quartic equation as above.

When $\delta_1 = 0$, the solution is more simple. For $n \leqq 13$, the maximum number of cusps is given by $[\kappa] \leqq \frac{1}{3}n(n-2)$. Hence, solving this inequality for n, we have

$$[n] \geqq 1 + \sqrt{3\kappa_1 + 1} \qquad \text{for} \qquad \kappa_1 \leqq 50.$$

For $n > 13$,

$$[\kappa] \leqq \tfrac{1}{4}n(n+3) - 4.$$

Solving for n we obtain

$$[n] \geqq \tfrac{1}{2}(\sqrt{16\kappa_1 + 73} - 3) \qquad \text{for} \qquad \kappa_1 > 50.$$

In the foregoing results, nothing has been said as to the relative positions of the double points. None of the curves can have more double points than has been assigned them, but in certain cases some of these double points may unite to form points of higher multiplicity. When the genus does not change, one additional independent absolute invariant is imposed on the curve for each increase in the multiplicity of a point by unity. For these curves, the only restriction on the coincidence of double points is that the total increase in multiplicity shall not exceed $3n + p - 9 - \kappa$. For $p < p_1$, this allows a certain number of coincidences, but for $p \geqq p_1$, all the double points assigned must be distinct for a proper curve.

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