ON THE RELATIVE CURVATURE OF TWO CURVES IN V_n*

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1. *Definitions of Geodesic Curvature.* In any space of n dimensions V_n whose first fundamental form is given by †

$$ds^2 = \sum_{ik} a_{ik} dx_i dx_k, \tag{1}$$

the geodesic curvature κ of a curve c at a point P may be defined in one of the two following ways:

(i) Draw the geodesic g tangent to c at P and on g and c lay off from P equal infinitesimal arc lengths δs; let Q and $\overline{Q}$ be the extremities of these arcs on c and g respectively; then ‡

$$\kappa = \lim_{Q \to P} \frac{2Q\overline{Q}}{(\delta s)^2}. \tag{2}$$

(ii) Consider an infinitesimal element $PQ = \delta s$ of c and the geodesic g having this element in common with c, i.e., as Q approaches P as a limit, c and g will approach tangency at P; the immediately following elements of c and g will not in general coincide but will form at Q an infinitesimal angle $\delta\omega$; then §

$$\kappa = \lim_{Q \to P} \frac{\delta\omega}{\delta s}. \tag{3}$$

Both these definitions lead to the same analytical expression

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† Throughout this paper all summations extend from 1 to n for the indicated subscripts.

‡ This definition is that given by Voss, MATHEMATISCHE ANNALEN, vol. 16. Cf. Bianchi, *Geometria differenziale*, 2d ed., vol. 1, p. 363.

§ This definition, or an equivalent one in terms of the parallelism of Levi-Civita, was given by the author in a paper, *Sulla curvatura geodetica delle linee appartenenti ad una varietà qualunque*, RENDICONTI ACCADEMIA DEI LINCEI, vol. 31 (1922).

for the geodesic curvature κ, viz.,

$$
(4)\qquad \kappa=\sum_{rt}a_{rt}\left[\frac{d^2x_r}{ds^2}+\sum_{ik}\left\{\begin{matrix} i\ k \\ r \end{matrix}\right\}\frac{dx_i}{ds}\frac{dx_k}{ds}\right] \times\left[\frac{d^2x_t}{ds^2}+\sum_{ik}\left\{\begin{matrix} i\ k \\ t \end{matrix}\right\}\frac{dx_i}{ds}\frac{dx_k}{ds}\right],
$$

where $\{{}^{i\,k}_{\ r}\}$ is the well known Christoffel symbol of the second kind.

It is the purpose of this note to generalize the above procedure by replacing the geodesic g in (i) by any other curve $\bar{c}$ tangent to c at P, and (ii) by any other curve $\bar{c}$ having an infinitesimal element PQ in common with c.

2. *Generalization of the First Definition.* We have any two curves c and $\bar{c}$ tangent at P and two equal infinitesimal arc lengths PQ and $P\bar{Q}$ $(=\delta s)$ on c and $\bar{c}$ respectively. Let the coordinates of P be x_r $(r = 1, 2, \cdots, n)$, those of Q and $\bar{Q}$ (developing in powers of δs) be, respectively,

$$
x_r + x_r'\delta s + \tfrac{1}{2}x_r''(\delta s)^2, \qquad (r = 1, 2, \cdots, n),
$$
$$
x_r + x_r'\delta s + \tfrac{1}{2}\bar{x}_r''(\delta s)^2, \qquad (r = 1, 2, \cdots, n),
$$

(disregarding infinitesimals of higher order than the second), where the direction of the common tangent at P is given by $x_r' = dx_r/ds$ $(r = 1, 2, \cdots, n)$, and where x_r'' and $\bar{x}_r''$ are the values of d^2x_r/ds^2 computed at P for c and $\bar{c}$ respectively. The differences of these coordinates are

$$
\tfrac{1}{2}(x_r'' - \bar{x}_r'')(\delta s)^2,
$$

and hence we have for the distance $Q\bar{Q}$ the expression

$$
(Q\bar{Q})^2 = \sum(a_{rt})_Q(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'')(\delta s)^4/4,
$$

where $(a_{rt})_Q$ represents the values of the coefficients a_{rt} at the point Q, i.e.,

$$
(5)\qquad (a_{rt})_Q = a_{rt} + \frac{da_{rt}}{ds}\delta s + \cdots.
$$

Therefore we have

$$
(6)\qquad \left[\lim_{Q\to P}\frac{2Q\bar{Q}}{(\delta s)^2}\right]^2 = \sum_{rt}a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'').
$$

3. *Generalization of the Second Definition.* We have any two curves c and $\bar{c}$ having an infinitesimal element $PQ = \delta s$ in common (i.e., as $Q \to P$, the curves c and $\bar{c}$ will approach tangency at P). The immediately following elements QR and $Q\bar{R}$ of c and $\bar{c}$ will form an infinitesimal angle $\delta\omega$ at Q. Let the direction PQ have for parameters $x_r' = dx_r/ds$ ($r = 1, 2, \cdots, n$), i.e.,

$$\sum_{rt} a_{rt} x_r' x_t' = 1. \tag{7}$$

The directions of QR and $Q\bar{R}$ may be expressed by the parameters $x_r' + x_r''\delta s$, $x_r' + \bar{x}_r''\delta s$ (disregarding infinitesimals of higher order than the first), bound by the relations

$$\left\{\begin{array}{l} \sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(x_t' + x_t''\delta s) = 1, \\ \sum_{rt}(a_{rt})_Q(x_r' + \bar{x}_r''\delta s)(x_t' + \bar{x}_t''\delta s) = 1. \end{array}\right. \tag{8}$$

The angle $\delta\omega$ between these two directions at Q is given by

$$\cos \delta\omega = \sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(x_t' + \bar{x}_t''\delta s).$$

Using the first identity (8), this becomes

$$\cos \delta\omega = 1 + \sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(\bar{x}_t'' - x_t'')\delta s. \tag{9}$$

Subtracting the identities (8), we have

$$2\sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(\bar{x}_t'' - x_t'')\delta s + \sum_{rt}(a_{rt})_Q(\bar{x}_r'' - x_r'')(\bar{x}_t'' - x_t'')(\delta s)^2 = 0,$$

and introducing this into (9), we find

$$\cos \delta\omega = 1 - \tfrac{1}{2}\sum_{rt}(a_{rt})_Q(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'')(\delta s)^2. \tag{10}$$

On the other hand,

$$\cos \delta\omega = 1 - \tfrac{1}{2}(\delta\omega)^2 + \cdots. \tag{11}$$

Comparing (10) and (11), we deduce that

$$\lim_{Q\to P}\left(\frac{\delta\omega}{\delta s}\right)^2 = \sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t''). \tag{12}$$

We note that the right members of (6) and (12) are identical.

4. *Relative Curvature.* The expression

$$\sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'') \tag{13}$$

has an interesting geometric interpretation. Let us set

$$A_r = \sum_{ik} \left\{ \begin{matrix} i\,k \\ r \end{matrix} \right\} x_r' x_k'. \tag{14}$$

We may then write (13) in the form

$$\begin{aligned} &\sum_{rt} a_{rt}[(x_r''+A_r)-(\bar{x}_r''+A_r)][(x_t''+A_t)-(\bar{x}_t''+A_t)] \\ &= \sum_{rt} a_{rt}(x_r''+A_r)(x_t''+A_t)+\sum_{rt} a_{rt}(\bar{x}_r''+A_r)(\bar{x}_t''+A_t) \\ &\qquad -2\sum_{rt} a_{rt}(x_r''+A_r)(\bar{x}_t''+A_t). \end{aligned} \tag{15}$$

We here introduce the geodesic curvature κ as given by (4), and the direction of the principal geodesic normal to a curve c at a point P as given by the parameters *

$$\mu^{(r)} = \frac{1}{\kappa}(x_r'' + A_r), \quad (r = 1, 2, \cdots, n), \tag{16}$$

so that

$$\sum_{rt} a_{rt}(x_r'' + A_r)(\bar{x}_t'' + A_t) = \kappa\cdot\bar{\kappa}\sum_{rt} a_{rt}\mu^{(r)}\bar{\mu}^{(t)} = \kappa\cdot\bar{\kappa}\cos\theta,$$

where θ is the angle between the principal geodesic normals to c and $\bar{c}$ at P. Then (15) or (13) takes the form

$$\kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta. \tag{17}$$

Finally, combining (6), (12) and (17), we have

$$\lim_{Q\to P} \frac{2Q\bar{Q}}{(\delta s)^2} = \lim_{Q\to P} \frac{\delta\omega}{\delta s} = \sqrt{\kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta}. \tag{18}$$

If, in the definitions (i) and (ii) of § 1, we replace the curve c and the tangent geodesic g by any two tangent curves c and $\bar{c}$, we shall say that the limiting expressions in (1) and (2) define the curvature of c relative to $\bar{c}$ or the *relative curvature* of c and $\bar{c}$. We may now state the following theorem.

Given any two curves c and $\bar{c}$ in V_n tangent at a point P. Let κ and $\bar{\kappa}$ be their respective geodesic curvatures and let θ be the angle between their principal geodesic normals at P. Then the relative curvature λ of c and $\bar{c}$ at P is given by

$$\lambda^2 = \kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta.$$

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* Bianchi, ibid., p. 364.