BOOKS ON FOURIER SERIES

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Second edition, revised throughout and enlarged. Vol. 1. Cambridge, the University Press, 1921. xv + 671 pp.

Introduction to the Theory of Fourier's Series and Integrals. By H. S. Carslaw. Second edition, completely revised. London, Macmillan and Company, 1921. xi + 323 pp.

The study of Fourier's series has exercised a profound influence upon the development of the theory of functions of a real variable.* Any one familiar with this influence and with the close relationship, both historical and inherent, between the two theories, will not be surprised that the two treatises mentioned above should have a considerable number of points in common. He will also not be surprised that in each case the plan of writing such a work had its origin in the study of Fourier's series in connection with their applications to problems of mathematical physics. The points of difference in scope and content between the two books are due in the main to the difference in the ultimate aim of the two writers. Professor Carslaw decided to write a book primarily for the worker in applied mathematics who has occasion to make use of Fourier's series and integrals; Professor Hobson elected to meet the needs of the pure mathematician whose work deals directly or indirectly with functions of a real variable.

A casual glance at the table of contents of Carslaw's work, however, will convince the informed reader that the needs of the pure mathematician of one generation are very apt to become the needs of the applied mathematician of the next generation. It is not so long ago that it would have been considered very unorthodox to include such topics as a discussion of Dedekind's theory of irrational numbers, the nature of uniform and non-uniform convergence, and a fairly complete treatment of the Riemann integral in a book intended for the applied mathematician. So, if the reviewer admits that Hobson's book is mainly for the pure mathematician, he does it with the mental reservation that the statement applies only to the present time and the immediate future. He does not agree with the implications contained in Hobson's statement on page 432 that the Riemann integral "will continue to be the basis upon which the practical applications of the Integral Calculus rest", but thinks it quite likely that at some future time, more or less distant, certain workers in applied mathematics may find their center of interest transferred from the Riemann integral to the Lebesgue integral, or some other integral still more general, just as at the present time some of those pursuing applied mathematics have found it desirable to change from a basis of euclidean geometry to a basis of non-euclidean geometry.

* Cf. E. B. Van Vleck, *The influence of Fourier's series upon the development of mathematics*, SCIENCE, new ser., vol. 39 (1914).

The rapidity with which the Lebesgue integral has become the standard type in present-day analysis is well exhibited by the difference in the relative emphasis placed on the Riemann and Lebesgue theories in the two editions of Hobson's book. In the first edition the latter theory was accorded a position of secondary importance; in the present edition it is placed on an equal basis, or if anything receives the greater stress. It is also illuminating in this connection to note that most of the new material included in the present edition is connected with advances concerned with or dependent on Lebesgue's theory.

As the reader will infer from the preceding paragraph, the second edition of Hobson's work takes due account of the great developments that have taken place during the past fifteen years in the theory of functions of a real variable and particularly in the theory of integration. Likewise Carslaw has not been content to rest on the laurels of his first edition, but has incorporated in his second edition many of the recent advances in the theory of Fourier's series and integrals that seem of special significance to the worker in applied mathematics. Thus both authors have aimed in the second edition as in the first to write the best possible book of the type selected on the basis of all the literature available at the time of writing. In both cases they have approached measurably near to the high ideal that they set for themselves.

Since the first editions of the two books are so widely known, the present review will not consider in great detail such portions of the second editions as are carried over from the first with relatively little change, but will concern itself mainly with the additions and the more important revisions that have been made.

Turning first to Hobson's work, we find that Chapter I is devoted to an adequate discussion of the system of real numbers, presenting both the Dedekind and the Cantor treatment of the irrationals. It is practically unchanged from the first edition except for the addition of a treatment of mathematical induction due to Padoa. The purpose of this discussion is to show that the principle referred to is a theorem which may be deduced from the properties of a simply infinite ascending aggregate by means of the principle of contradiction. This is essentially a refutation of Poincaré's contention that the principle of mathematical induction is a special characteristic of mathematical reasoning and cannot be reduced to the principle of contradiction.

Chapters II and III, which deal respectively with the descriptive properties and the metric properties of sets of points, constitute the revised form of the second chapter of the first edition. The material contained in this chapter has been much expanded in order to prepare the way for the more extensive treatment of Lebesgue integrals and their generalizations, and in numerous instances the form of the exposition has been considerably changed. One of the most extensive alterations of this sort is in connection with the treatment of non-linear sets of points. In the first edition this treatment was postponed until the exposition of the properties of linear sets had been completed. In the present edition the notion of sets of points in two or more dimensions is introduced early in Chapter II, and the more important theorems with regard to linear sets are extended at once to sets of several dimensions after they have been proved for the case of one dimension. This change has both advantages and disadvantages. It does serve to focus the attention on the generality of the theorems involved, but on the other hand it interrupts the continuity of the argument. Opinions will probably differ as to whether or not it constitutes an improvement.

Chapter IV, which deals with transfinite numbers and order types, corresponds to Chapter III in the first edition. Relatively little change has been made in the material presented, except that the discussion of certain disputed points in the theory of assemblages, which concludes the chapter, has been revised and expanded.

Chapter V, which deals with general properties of functions of a real variable and which corresponds to Chapter IV of the first edition, contains a considerable amount of new material. This is quite natural in view of the many new developments in function theory proper during the period from 1907 to 1920. Some of this added material was available in 1907, but in most of these cases it was probably the light shed by later developments which changed the author's opinion as to its relative importance. Among the new topics discussed may be mentioned in particular the following: absolute continuity, approximate continuity, the symmetry of functional limits, functions of a variable aggregate. In connection with the latter topic, the notion of a general variable is briefly described and some of the recent developments centering about this idea are mentioned. In view of the prominent rôle of the general variable in much recent work in analysis of fundamental importance, a fuller account of it would seem desirable, even in a work devoted primarily to a very complete treatment of classical analysis.

Chapters VI, VII, and VIII are devoted to the theory of integration, and correspond to Chapter V and a small portion of Chapter VI of the first edition. This part of the work has had the greatest expansion, and this expansion is mostly in connection with the Lebesgue integral and some of its generalizations. As pointed out before, one of the chief characteristics of the present edition of the book, as contrasted with the first, is the shift in emphasis from the Riemann theory to the Lebesgue theory. The tendency in the later edition is to make this latter theory the central topic in integration and group the other developments about it. Thus, although the Riemann integral, on account of its historical priority and the somewhat greater concreteness of the geometric ideas underlying its definition, is naturally treated first, some of the important properties of the definite integral, such as the mean value theorems, are proved for the first time after the introduction of Lebesgue integrals.

In addition to a complete account of the Riemann and Lebesgue integrals, the three chapters devoted to integration contain more or less extended treatments of other types of integrals, such as the Riemann-Stieltjes integral, the Lebesgue-Stieltjes integral, Hellinger's integrals, the Denjoy integral, and the Young integral. Since the theory of integration is still in a stage of active development, it is rather too much to expect a treatise such as Professor's Hobson's, which naturally aims to include only such researches as have reached a state of comparative completeness, to give an entirely adequate picture of the present state of the theory. Moreover, some authorities might disagree with him as to the relative emphasis he accords to the various generalizations of the notion of integration. But the reader who has perused with care the 230 odd pages devoted to this highly important topic will have a fairly satisfactory idea of the wide extension of its scope involved in the newer definitions and the much greater power of these definitions in dealing with the applications of the theory.

The reviewer has earlier expressed his high opinion of the book as a whole. It is hardly to be expected that a work of this size and scope should be free from minor defects; there are certain of these that it seems worth while to indicate. Hobson's references to the literature would have been much more valuable to the reader if he had uniformly included dates of publication in the case of all important memoirs. The almost uniform omission of such dates is hardly excusable in an account of a theory where the works of various writers have so many interrelations and the reader naturally wishes to orient himself as to the order of ideas and the reaction of one line of thought upon another. The use of the symbol $\sim$ to indicate approach to a limit also seems unwise to the reviewer. While this symbol has had some connection with approach to a limit when used in the case of asymptotic series, it has generally been used to represent correspondence in a broader sense. The symbol $\rightarrow$ is much more suggestive of the underlying notions and has been widely adopted by analysts in the time that has elapsed since its introduction.* It does not appear likely that Hobson's notation will supplant it.

Turning now to Carslaw's book, we find that the first volume of the second edition exhibits an expansion over the corresponding material in the first edition that is even greater proportionally than in the case of Hobson's book. This will not be surprising to those familiar with recent developments in the theory of Fourier's series and integrals, for many of these developments have been of special significance to those primarily interested in the applications of the theory. The new material added by Carslaw is not exclusively composed of such new developments, however. A considerable portion of it is devoted to a more elaborate development of certain topics of pure analysis that are of fundamental importance for those who seek to use in any rational manner such tools as Fourier's series and integrals in the field of applied mathematics.

The book begins as before with an interesting historical introduction, which in the present edition has been somewhat revised and expanded. Chapter I, devoted to an exposition of the system of real numbers, and Chapter II, dealing with the more fundamental properties of infinite sequences and series, contain substantially the same material as in the

* It may be worthy of note in this connection that this BULLETIN has adopted this notation from the beginning of the present volume. (The editors.)

first edition, though they have been considerably revised. Chapter III of the present edition is new and contains a development of the notions of function and limit and some of the more fundamental properties of functions of one or more variables. The addition of this chapter is in line with a suggestion by the late J. E. Wright in the course of his review of the first edition in this BULLETIN.

Chapter IV deals with the same material as the first part of Chapter IV in the first edition, but the material has been much expanded and revised. An adequate treatment of the Riemann integral and its more important properties has been included in this chapter. Chapter V is the original Chapter III in revised form and with a considerable amount of new material at the end. This includes a proof of Bromwich's theorem on convergence factors which is of importance in many applications of Fourier's series and allied developments, and a discussion of the integration of infinite series. Chapter VI contains the remaining portion of Chapter IV, namely, the treatment of definite integrals involving a parameter. This portion also has been revised and expanded.

Chapter VII corresponds to the original Chapter V and is devoted to the discussion of the more fundamental theorems with regard to the convergence and summability (C1) of the Fourier development of an arbitrary function. Among the new material added may be mentioned the proof of the convergence of Fourier's series by means of Fejér's theorem on the summability (C1) and a general theorem with regard to series summable (C1) due to G. H. Hardy; also a discussion of Poisson's integral.

Chapter VIII, dealing mainly with the uniform convergence of Fourier's series and the validity of term by term differentiation and integration of such series, and Chapter IX, devoted to the consideration of the approximation curves and Gibb's phenomenon, correspond to Chapters VI and VII of the original edition. The material on Gibb's phenomenon is new and is founded mainly on Bôcher's discussion in the ANNALS OF MATHEMATICS and the further treatment of the subject by the author in the AMERICAN JOURNAL OF MATHEMATICS.

Chapter X, which deals with Fourier's integrals, corresponds to the original Chapter VIII. In its present form it includes recent extensions of the theory of these integrals due to Pringsheim. Appendix I, which is new, gives a brief account of certain methods of harmonic analysis that are useful when the function to be developed is not given in analytic form but only by means of a curve obtained from observations. Appendix II contains the bibliography of Appendix A in the first edition, brought up to date. This bringing up to date involves the addition of seven and a half pages of titles for the period 1906–1920.

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