EXTENSIONS OF DIRICHLET MULTIPLICATION AND DEDEKIND INVERSION*

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1. *Introduction.* With Landau,† let us define *Dirichlet multiplication* to be the following formal process of combining two sequences α_n, β_n $(n = 1, 2, \cdots)$ to form a third sequence γ_n, $\gamma_n = \Sigma\alpha_l\beta_m$, the sum extending to all integers l, $m > 0$ such that $lm = n$. Henceforth, unless otherwise stated, we assume the four rational algebraic operations to be purely formal.‡ When any of these operations in relation to a given system of elements have special interpretations, examples of which are noted in a moment, they will be called *specific*. Restating Landau's definition, let us denote by α_n, β_n $(n = 1, 2, \cdots)$ two classes of elements which are such that the product of any element of one class by an element of the other has a unique significance, and likewise for any sum of such formal products. Then the *Dirichlet product* of these classes is the class $(\alpha, \beta)_n$ $(n = 1, 2, \cdots)$, where $(\alpha, \beta)_n = \Sigma\alpha_l\beta_m$ $(lm = n)$.

When the α's and β's are known, the process of determining the φ's from the relation $(\alpha, \varphi)_n = \beta_n$ for n an arbitrary integer > 0 will be called *Dedekind inversion*. It will be shown that the extension of this which we have in view includes Dedekind's inversion in the theory of numbers.§

The interpretations of the general theorems in a specific case will depend only upon the meanings assigned to the elements and to the rational operations upon those elements. Thus, if the elements are numbers and the rational operations are as in arithmetic, the interpretation is obvious; if the elements are classes, multiplication and addition are logical,

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† Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 2, p. 671. We replace his series by sequences.

‡ As in the customary definitions of an abstract field, cf. Dickson, *Linear Groups*, pp. 5–6. When some of the operations in a field are specific, the field will be called special. See also § 6.

§ The inversion is usually attributed to Dedekind, although it was published simultaneously by Liouville. Cf. Dickson's *History*, vol.1, p. 441.

their inverses are not defined, and the theorems apply only to all sums and products derived from products which are themselves obtained by multiplications from a primary set;* if the elements are single-valued functions (or if they are associates, as presently defined), the important cases for the arithmetic of integers or of ideals, the interpretations of the rational operations are as developed later in this paper. By means of the last a great mass of the material which is summarized in Chapters V, X, XIX of volume I of Dickson's *History of the Theory of Numbers*, and which is not most naturally derived from the elements of elliptic functions, can be simultaneously restated in much condensed form, unified, proved almost at a glance, and extended in many directions. This, the most immediate application of what follows, will be discussed elsewhere.†

2. *Definitions*. (i) Let A' be a class of n' independent elements x_a ($a = 1, 2, \cdots, n'$) subject only to the commutative and associative laws of multiplication. By definition the zero powers of all elements in A' are equal, and each is equal to the multiplicative unit e, thus $x_a^0 = e$ ($a = 1, 2, \cdots, n'$), $e^2 = e$, etc.; and n' may be finite or infinite. From the elements of A' we form the class A of all products of the type $x_a^\alpha x_b^\beta \cdots x_c^\gamma \equiv \Pi x_a^\alpha$, where $a, b, \cdots, c$ are different members of the set $1, 2, \cdots, n'$, and $\alpha, \beta, \cdots, \gamma$ are integers $\geqq 0$, and denote this typical product by x. Products differing only by unit factors (powers of e) are equal. Hence if n' is finite, A may consist of either a finite or infinite number n of distinct products. The class A evidently includes all the members of A'. The members of A' are called the *primary elements* of A. Elements of A' are called the *derived elements*, and x is a typical derived element. From these definitions we may con-

* Subtraction and division are omitted for the reasons given by Whitehead, *Universal Algebra*, vol. 1, p. 82.

† Another application of importance for arithmetic, which has been developed in detail, is to the case in which the elements are sequences. A brief account of results for elliptic functions cited above which involve Bernoullian functions appeared recently in the MESSENGER OF MATHEMATICS; generalizations for certain of the other divisor theorems depending on elliptic functions will be published later.

sider the primary elements to be independent generators of an abelian group A whose identity is e. The order of the group is n, which may be finite or infinite. When A' is the class of prime numbers and multiplication is as in arithmetic, the resulting theory is called the numerical case.

(ii) An *associate* of an element in A is anything that has a unique significance in terms of elements of A, when the element, called the *argument* of the associate, is assigned. It is assumed that associates admit of unambiguous combination by formal rational operations (§ 1) with associates. If A be replaced throughout in what precedes by A', the class of primary elements of A, there is defined a primary associate. When the argument is a derived element, the associate is called derived. This distinction is made because all that follows is based upon associates that need exist only when their arguments are primary elements; derived associates are constructed (in (iv)) from the primaries.*

(iii) The notation being as in (i), the $\nu(x) = (\alpha + 1)(\beta + 1) \cdots (\gamma + 1)$ derived elements x' given by

$$x' = x_a^{\alpha'} x_b^{\beta'} \cdots x_c^{\gamma'} \quad (0 \leqq \alpha' \leqq \alpha,\ 0 \leqq \beta' \leqq \beta,\ \cdots,\ 0 \leqq \gamma' \leqq \gamma)$$

are by definition the divisors of the typical derived element x. In the product $x \equiv \Pi x_a^{\alpha}$ of powers of distinct primary elements, corresponding Latin and Greek letters (a, α), (b, β), $\cdots$, (c, γ) are associated with a given primary element, thus

* Since it seems not to be customary except in mathematical logic (*Principia Mathematica*, vol. 1, pp. 15–18) to speak of either values or functions apart from a numerical context, we use *associate* instead of *single-valued function*, although the latter would be justified by the current use of *propositional function*. For special interpretations of the several parts of (ii) the definition of associate degenerates as follows to that of single-valued function of an integer > 0. Let $\varphi(x)$ be an associate having the integral non-zero argument x. Give the parts of (ii) the following specific meanings, which obviously are legitimate: the rational operations are as in ordinary algebra; $A \equiv$ the class of integers > 0; the phrase "when the element is assigned" $\equiv$ *when x lies in the interval (a, b) of A;* "has a unique significance in terms of elements of A" $\equiv$ *has a single definite value corresponding to each x $(a \leqq x \leqq b)$, no matter in which form the correspondence is specified.* With these interpretations, $\varphi(x)$ is a single-valued function of x in the interval (a, b). Cf. Dirichlet's definition (*Werke*, vol. 1, p. 135), also Hobson, *Theory of Functions of a Real Variable*, 1st edition, p. 216) for the analogous definition when x, (a, b) are continuous.

$x_a^{\alpha}, x_b^{\beta}, \cdots, x_c^{\gamma}$. This convention is maintained throughout when the exponent is an arbitrary positive integer.

(iv) With each positive integral or zero power x_d^i of any primary element of x_d are associated specific primary associates $\varphi_i(x_d), \psi_i(x_d), \cdots, \mu_i(x_d), \cdots$ of that element, the suffix i of the primary associate being equal in each case to the exponent of the power. For example the φ-primary associates of x_a^0, $x_a^1, x_a^2, \cdots$ are $\varphi_0(x_a), \varphi_1(x_a), \varphi_2(x_a), \cdots$. By definition all primary associates of e are equal, and each is denoted by ϵ. Hence since by (i) $x_a^0 = x_b^0 = \cdots e$, we have $\varphi_0(x_a) = \varphi_0(x_b) = \cdots = \epsilon$; $\mu_0(x_a) = \mu_0(x_b) = \cdots = \epsilon$, etc. We call ϵ the *unit associate. It is assumed that, with respect to any associate,* ϵ *has all the multiplicative properties of unity in arithmetic.* Beyond the trivial one that they both have the same argument x_a, it is not assumed (without explicit statement of the assumption) that there is any relation whatever between $\varphi_m(x_a)$, and $\varphi_n(x_a)$ (m, n integers $\geqq 0$, $m \neq n$); nor is it assumed that primary associates having different suffixes are necessarily distinct.*

From the definition in (ii), it follows that, for $x = \Pi x_a^{\alpha}$ as in (i),

$$\Pi\varphi_{\alpha}(x_a) \equiv \varphi_{\alpha}(x_a)\varphi_{\beta}(x_b) \cdots \varphi_{\gamma}(x_c)$$

is an associate of x. We call this a *derived associate*, and we shall denote it by φ_x. The derived associate $\varphi_{x'}$ of the divisor x' of x is called an *associate divisor* of φ_x. Similarly, starting with the primary associates $\psi_{\alpha}(x_a), \psi_{\beta}(x_b), \cdots, \psi_{\gamma}(x_c)$, we define $1 + \nu(x)$ derived associates $\psi_x, \psi_{x'}$; and so on for $\chi_x, \chi_{x'}$; $\lambda_x, \lambda_{x'}$; $\cdots$; $\mu_x, \mu_{x'}$. Thus $\mu_x = \Pi\mu_{\alpha}(x_a)$. Henceforth we shall drop the adjectives *primary* and *derived*, and refer only to associates, since the notations $\varphi_{\alpha}(x_a)$, φ_x suffice without qualifications.

3. *Relation with Group Characters*. Before proceeding, we emphasize the remarks made in § 1 regarding the interpretations of results. This has particular reference to the assump-

* For example in the special case of associate $\equiv$ single-valued function, we may have $\varphi_{\alpha}(x_a) = \sin \pi x_a$, $\varphi_{\beta}(x_b) = \sin \pi x_b$; namely each is the sine of π times its argument.

tion in § 2 (ii), to which we shall add some comments in § 6. The development is necessarily abstract, because in the applications it is essential that proofs be based upon considerations that are independent of convergence, or indeed of any infinite process, and because each general theorem is susceptible of many specific interpretations. (Cf. § 1.) It may assist the reader to consult § 8 occasionally. We insist, however, that nothing in § 8 is assumed in any of the proofs, all of which are immediate consequences of the definitions, except that those after Theorem XIV depend also upon the assumption that an arithmetic system exists. This assumption is satisfied by the class of all rational functions in any number n of independent variables with integral coefficients, which for $n = 1, 2$ is the important case arithmetically. We have assumed that for each primary element there exists at least one associate. This can be satisfied, for example, by taking the associate equal to the element. Hence the theory as developed relates to objects that exist.

In addition to possible interpretations mentioned in § 1 we note one here which is of particular interest.* If we impose the restrictions $f_2(x_a) = [f_1(x_a)]^2$ $(f = \varphi, \psi, \chi, \cdots, \lambda, \mu, \cdots)$ upon the associates, they can be interpreted as group characters.†

4. *Extended Dirichlet Product.* If the derived elements x_1, x_2 (§ 2 (i)) have in common only unit factors (powers of e), they are called *relatively distinct* (the analogue of *relatively prime*).

THEOREM I. *If x_1, x_2 are relatively distinct,* $\varphi_{x_1}\varphi_{x_2} = \varphi_{x_1x_2}$.

Next, let x_1, x_2 be *any* pair of divisors (§ 2 (iii)) of x that are such that $x_1x_2 = x$; form $\varphi_{x_1}\psi_{x_2}$ for each pair (x_1, x_2); take the sum of all such products for all possible pairs (x_1, x_2); and denote that sum by $(\varphi, \psi)_x$. It is obvious, by § 2 (ii), (iv), that this sum is an associate of x, and hence the process

* This was pointed out by Professor Dickson. I had not noticed it, since I was developing the theory with reference to its applications to sequences, classes, and the multiplicative properties of numbers and ideals.

† For the case of abelian groups, cf. Weber, *Algebra*, 2d edition, vol. 2, p. 43; for an exposition of the general case, cf. Dickson, ANNALS OF MATHEMATICS, (2), vol. 4, 1902, pp. 25–49.

may be repeated for $(\varphi, \psi)_x$ and χ_x as initial associates instead of φ_x, ψ_x; denote the result by $((\varphi, \psi), \chi)_x$, or briefly by $(\varphi, \psi, \chi)_x$. Continuing thus, starting with m associates $\varphi_x, \psi_x, \chi_x, \cdots, \lambda_x, \mu_x$, taken in this order, we reach finally an associate $(\varphi, \psi, \chi, \cdots, \lambda, \mu)_x$ of x, which is called the *Dirichlet product* of the original m associates, and the original m associates are called the *primitive divisors* of the extended product.

THEOREM II. *The extended product* $(\varphi, \psi, \cdots, \mu)_x$ *is a symmetric function of* $\varphi, \psi, \cdots, \mu$. *In other words, extended multiplication is associative and commutative.*

This being a result of great power in applications, we shall examine it more closely. Let $x_1, x_2, \cdots, x_m$ denote divisors of x such that $x = x_1 x_2 \cdots x_m$. Then we have

$$(\varphi, \psi, \cdots, \mu)_x = \Sigma \varphi_{x_1} \psi_{x_2} \cdots \mu_{x_m},$$

where the sum extends to all sets $(x_1, x_2, \cdots, x_m)$. The m functions in the typical term on the right are associate divisors (§ 2 (iv)) of $\varphi_x, \psi_x, \cdots, \mu_x$, respectively. From the mode of formation it is evident that the sum (i.e., the extended Dirichlet product) is invariant under permutations of some or all of $x_1, x_2, x_3, \cdots, x_m$. This sum is an associate of x, say $F_x^{(m)}$, which is uniquely determined whenever x and the primitive divisors $\varphi_x, \psi_x, \cdots, \mu_x$ of $(\varphi, \psi, \cdots, \mu)_x \equiv F_x^{(m)}$ are assigned; in particular it is uniquely known when x is replaced by any one of its divisors x'. In the same way, starting with any n associates $\alpha_x, \beta_x, \cdots, \gamma_x$, we arrive at $G_x^{(n)}$, and continuing thus we obtain a system of r such extended products $F_x^{(m)}, G_x^{(n)}, \cdots, H_x^{(t)}$, there being r letters $m, n, \cdots, t$. Let $m + n + \cdots + t = s$, and suppose that the s associates upon which the r extended products are constructed are $\alpha_x, \beta_x, \cdots, \omega_x$, these being not necessarily distinct. Then we have

$$(F^{(m)}, G^{(n)}, \cdots, H^{(t)})_x = (\alpha, \beta, \cdots, \omega)_x.$$

Except when $m = n = \cdots = t = 1$, the primitive divisors $F_x^{(m)}, G_x^{(n)}, \cdots, H_x^{(t)}$ and $\alpha_x, \beta_x, \cdots, \omega_x$ of these extended products are not the same. On the left, the extended product refers to resolutions of x into r divisors which in turn are resolved into $m, n, \cdots, t$ divisors respectively, the resolutions in both cases being all possible of their respective kinds; on the right, the resolution is into all possible sets of s divisors. The identity of the formal sums for these distinct methods of resolving x is the interpretation of the associative and commutative properties of extended multiplication. The significance of Theorem II in the numerical case was stated in a former paper.*

In order to see in the numerical case that $(\varphi, \psi)_x$ is equivalent to $(\alpha, \beta)_n$ of § 1 when φ_x, ψ_x, and the α's and β's are single-valued functions of integers, take as primary elements the natural primes, choose $\varphi_\rho(x_r) = \varphi'(x_r{}^\rho)$ where φ' is single-valued and such that $\varphi'(mn) = \varphi'(m)\varphi'(n)$ when m and n are relatively prime integers greater than zero. In the same way, define $\psi_\rho(x_r) = \psi'(x_r{}^\rho)$, and finally write $\varphi'(x)$, $\psi'(x) \equiv \varphi_x', \psi_x'$. Then for α, β, n in the numerical case equal respectively φ', ψ', x, as above, we have $(\varphi', \psi')_x = (\alpha, \beta)_n$.

Next, the sum of the associates $\chi_x, \cdots, \lambda_x, \mu_x$ is obviously an associate, σ_x, of x, and it is easily seen that

$$(\varphi, \sigma)_x = (\varphi, \chi)_x + \cdots + (\varphi, \lambda)_x + (\varphi, \mu)_x.$$

THEOREM III. *Extended Dirichlet multiplication is distributive with respect to (formal) addition.*

5. *Extended Dedekind Inversion.* The unit associate ϵ was defined in § 2 (iv). We now introduce the *associate unit* $\eta_x \equiv \eta$, whose definition is as follows: $\eta_x = 0$ (the multiplicative zero) when $x \neq e$ (the multiplicative unit of A); $\eta_x = \epsilon$ (the unit associate) when $x = e$.

THEOREM IV. *For each associate φ_x there exists a unique associate φ_x' such that $(\varphi, \varphi')_x = \eta$.*

We shall call φ_x' the *reciprocal* of φ_x, and, by an obvious algebraic analogy, we shall write $\varphi_x' = \eta/\varphi_x$, $\varphi_x = \eta/\varphi_x'$. We have at once, ψ_x being any associate, $(\psi, \eta)_x = \psi_x$; and hence we have the following theorem.

THEOREM V. *The unit with respect to extended Dirichlet multiplication is η.*

To prove Theorem IV, we shall exhibit φ_x' explicitly in terms

* This BULLETIN, vol. 27, p. 274, § 2.

of φ-associates. We observe first that if $x = \Pi x_a^{\alpha}$ as usual, Theorem I enables us to state that φ_x' is equal to the formal product of the associates φ_ξ' for $\xi = x_a^{\alpha}, x_b^{\beta}, \cdots, x_c^{\gamma}$. Hence it is sufficient to determine φ_x' when $x = x_a^{\alpha}$, $\alpha > 0$. By the convention of § 2 (iv), the φ-primary associates of $x_a^1, x_a^2, \cdots, x_a^{\alpha}$ are respectively $\varphi_1(x_a), \varphi_2(x_a), \cdots, \varphi_\alpha(x_a)$. For $x = x_a^{\alpha}$, write φ_x' in the form $\varphi_\alpha'(x_a)$. Then it can be verified without difficulty that $\varphi_\alpha'(x_x)$ is equal to the determinant

$$\left(-\frac{1}{\epsilon}\right)^{\alpha}\begin{vmatrix} \varphi_1(x_a) & \epsilon & 0 & \cdots & 0 \\ \varphi_2(x_a) & \varphi_1(x_a) & \epsilon & \cdots & 0 \\ \varphi_3(x_a) & \varphi_2(x_a) & \varphi_1(x_a) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{\alpha-1}(x_a) & \varphi_{\alpha-2}(x_a) & \varphi_{\alpha-3}(x_a) & \cdots & \epsilon \\ \varphi_\alpha(x_a) & \varphi_{\alpha-1}(x_a) & \varphi_{\alpha-2}(x_a) & \cdots & \varphi_1(x_a) \end{vmatrix}.$$

It is easy to prove this also by mathematical induction from $\alpha - 1$ to α, $\alpha > 1$. Next, φ_x' is written down as the formal product of $\varphi_\alpha'(x_a), \varphi_\beta'(x_b), \cdots, \varphi_\gamma'(x_c)$, which are obtained from the above on replacing (α, a) in turn by $(\alpha, a), (\beta, b), \cdots, (\gamma, c)$. It may be mentioned that in practice the explicit form of φ_x' is not often required; its existence is the important fact. Henceforth we indicate the reciprocal of a given associate by accenting the symbol of the latter as above.

THEOREM VI. *There is a unique* φ_x *satisfying the relation* $(\varphi, \mu)_x = \omega_x$ *when* μ_x, ω_x *are given:* $\varphi_x = (\omega, \mu')_x$.

For, from the given relation, $(\omega, \mu')_x = ((\varphi, \mu), \mu')_x = (\varphi, \mu, \mu')_x = (\varphi, (\mu, \mu'))_x = (\varphi, \eta)_x = \varphi_x$. In the same way, or as a corollary, we have the general case of such inversion:

THEOREM VII. *If* $(\varphi, \psi, \cdots, \lambda, \chi, \sigma, \cdots, \mu)_x = (\pi, \tau, \cdots, \omega)_x$, *then* $(\chi, \sigma, \cdots, \mu)_x = (\pi, \tau, \cdots, \omega, \varphi', \psi', \cdots, \lambda')_x$.

An interpretation in the numerical case of Theorem VI is as given in § 3 of the paper cited in § 4. For let the associates be single-valued functions of their arguments. Then if $\varphi_i(n) = 1$ for $n > 0$ an integer in the formula $\varphi_i = \varphi_k \varphi_i'$ stated in that paper, $\varphi_i'(n)$ is Möbius' $\mu(n)$, and the result is Dedekind's inversion, which thus appears as a very special consequence of the theorem in the numerical case.

6. *The Special Field of all Associates.* For greater clearness, before stating the next theorem, we recall the significance of the terms involved. For a rigorous discussion of the relevant logic we must refer the reader to *Principia Mathematica*, volume 1, pages 15–27, 85–88, 143–156, 260–268, 278–291. The term *real variable* is used as defined on page 18 of that work. The elements of an abstract field are real variables; theorems relating to an abstract field are assertions of propositional functions (ibid., pp. 15, 19). Our fundamental assumption in § 2 (ii) is "associates admit of unambiguous combination by formal rational operations with associates." Thus, in particular, it is assumed that in the abstract field φ_ξ ($\xi =$ any element of A, $\varphi_\xi =$ any associate of ξ), statements of identity between formal sums, differences, products, and quotients of elements (associates) in the field are propositional functions. On referring to the definitions of *extended Dirichlet product* and *reciprocal,* we see that only formal sums and products of associates are involved in the former, and only formal division by a power of ϵ, formal sums, differences and products of associates are involved in the latter. Hence statements of identity between Dirichlet products and reciprocals are propositional functions; and Theorem VIII is (*Principia Mathematica,* p. 18) an "ambiguous assertion" for every such propositional function in the abstract field of associates. These propositional functions in every case, *when Dirichlet products and reciprocals are written in full,* reduce to identities in *formal* addition $(+)$, subtraction $(-)$, multiplication (ab, etc.), division (a/b), when, as assumed, these operations have with respect to associates $a, b, \cdots$ their usual formal properties, $a + a - a = a$, $ab = ba$, $a(b + c) = ab + ac$, etc.

When in Theorem VIII special meanings are assigned to the associates and operations, it becomes a proposition concerning a special class of associates. An important case is that in which the associates are single-valued functions. To formal addition in this case we give the following meaning: The elements x are those of A, § 2 (i). If φ_x has a unique value

$\bar{\varphi}_x$ for each element x, and ψ_x a unique value $\bar{\psi}_x$ for each x, then the single-valued function having for each element x the value $\bar{\varphi}_x + \bar{\psi}_x$ is called the sum $\varphi_x + \psi_x$ of φ_x and ψ_x. Similarly for formal subtraction. For formal multiplication in this case, the single-valued function having for each element x the value $\bar{\varphi}_x\bar{\psi}_x$ is called the *product* $\varphi_x\psi_x$ of φ_x and ψ_x; and similarly for division. And so in each special case of Theorem VIII, the indicated *specific* operations of Dirichlet multiplication as defined in § 4 and division (extended Dedekind inversion) as in Theorem VII, may be said to have been broken down into their equivalents in terms of formal operations, to which are given special interpretations according to the special meanings assigned to the associates. Combining Theorems II, III, V, VII we have now the following powerful result.

THEOREM VIII. *All associates of the elements of an abelian group constitute a special field in which addition and subtraction are formal, multiplication is extended Dirichlet multiplication and division is extended Dedekind inversion.*

7. *Extended Cauchy Product.* Although Theorem VIII is sufficient for most purposes, it is of interest in many applications to proceed from it to an arithmetic classification of the results. This is not always feasible, but a majority of the known theorems on the arithmetic functions mentioned in the introduction, and the entire body of results of which these are but a part, can be so classified upon the following simple basis.

Restating Landau's definition (loc. cit., p. 670) of Cauchy multiplication, we call the class $[\alpha, \beta]_s$ $(s = 0, 1, \cdots)$ the *Cauchy product* of the classes α_s, β_s $(s = 0, 1, \cdots)$, when $[\alpha, \beta]_s = \sum_{r=0}^{s}\alpha_r\beta_{s-r}$. Repeating the process for $[\alpha, \beta]_s$ and γ_s $(s = 0, 1, \cdots)$, we form the class $[\alpha, \beta, \gamma]_s$ $(s = 0, 1, \cdots)$, and so on, getting finally, for any number of classes $\alpha_s, \beta_s, \cdots, \delta_s$ $(s = 0, 1, \cdots)$, the Cauchy product of all of those classes, $[\alpha, \beta, \cdots, \delta]_s$ $(s = 0, 1, \cdots)$. We call $\alpha_s, \beta_s, \cdots, \delta_s$ the factors of $[\alpha, \beta, \cdots, \delta]_s$. Clearly $[\alpha, \beta, \cdots, \delta]_s$ is invariant under permutation of some or all of its factors (this is equivalent to permuting $\alpha, \beta, \cdots, \delta$ within $[\alpha, \beta, \cdots, \delta]_s$); and if $\alpha_s + \beta_s = \gamma_s$ $(s = 0, 1, \cdots)$, then $[\gamma, \delta]_s = [\alpha, \delta]_s + [\beta, \delta]_s$. That is, this multiplication is distributive.

THEOREM IX. *Extended Cauchy multiplication is associative, commutative, and with respect to formal addition, distributive.*

The important case arithmetically is that in which the factors of a Cauchy product are single-valued functions. Let $x = x_a^\alpha x_b^\beta \cdots x_c^\gamma$ as in preceding sections. Then the following theorem is an important consequence of the definitions.

THEOREM X. *The sign* Π *of formal multiplication extending to all* $(\rho, r) = (\alpha, a), (\beta, b), \cdots, (\gamma, c)$,

$$(\varphi, \psi)_x = \Pi[\varphi(x_r), \psi(x_r)]_\rho \equiv \Pi[\sum_{s=0}^{\rho} \varphi_s(x_r)\psi_{\rho-s}(x_r)].$$

By repetition, we reach the following general result:

THEOREM XI.

$$(\varphi, \psi, \cdots, \mu)_x = \Pi[\varphi(x_r), \psi(x_r), \cdots, \mu(x_r)]_\rho.$$

From Theorems IV and X, we arrive at the inverse process, which we shall call *extended Cauchy division*, in the following theorems.

THEOREM XII. $[\varphi(x_a), \varphi'(x_a)]_\alpha = \eta$;

THEOREM XIII. *The first of the following relations implies the second,*

$$[\varphi(x_a), \cdots, \lambda(x_a), \chi(x_a), \cdots, \mu(x_a)]_\alpha = [\pi(x_a), \cdots, \omega(x_a)]_\alpha,$$
$$[\chi(x_a), \cdots, \mu(x_a)] = [\pi(x_a), \cdots, \omega(x_a), \varphi'(x_a), \cdots, \lambda'(x_a)]_\alpha.$$

Theorem XIII is analogous to Theorem VII, from which, in conjunction with Theorem X for $x \equiv x_a^\alpha$, it can easily be proved. Combining these results we have the analogue of Theorem VIII.

THEOREM XIV. *All the associates of the elements of an abelian group constitute a special field in which addition and subtraction are formal, while multiplication and division are extended Cauchy multiplication and its inverse.*

We shall call an abelian group of finite or infinite order which has a unique basis,* *an arithmetic system.* This name

* Cf. Weber, loc. cit., p. 33.

is chosen because such a group has the characteristic distinguishing arithmetic from algebra, that is, a unique factorization law. Arithmetic systems are abstract or special; in the former multiplication is abstract, in the latter, specific. Let us assume as a hypothesis that the class of all associates is an arithmetic system under extended Cauchy multiplication. Let us call the elements of the basis the *prime associates of the system:* a prime associate is the extended Cauchy product of no pair of prime associates each distinct from ϵ, η. From Theorem XI, we have then the following theorem.

THEOREM XV. *The specific arithmetic system of associates in which multiplication is extended Cauchy multiplication is also a specific arithmetic system in which multiplication is extended Dirichlet multiplication.*

8. *Connection with Formal Expansions.* With each element x_i $(i = 0, 1, \cdots)$ of an abelian group (or of an arithmetic system) let us associate an umbra z_i, and let us write

$$\bar{\varphi}(x, z) = \Pi_i[\varphi_0(x_i)z_i^0 + \varphi_1(x_i)z_i^1 + \cdots + \varphi_n(x_i)z_i^n + \cdots]$$

and similarly for $\bar{\psi}(x, z), \cdots, \bar{\mu}(x, z)$. Then the coefficients of $z_a^\alpha z_b^\beta \cdots z_c^\gamma$ in the formal developments of $\bar{\varphi}(x, z)$, $\bar{\varphi}(x, z)\bar{\psi}(x, z) \cdots \bar{\mu}(x, z)$, and the reciprocal of the latter, are, respectively, φ_x, $(\varphi, \psi, \cdots, \mu)_x$, $\eta/(\varphi, \psi, \cdots, \mu)_x$.

9. *Note.* We have nowhere intended to imply that the processes of this paper are more than *extensions* of Dirichlet multiplication and of Dedekind inversion. In fact it can be shown that the processes of this paper and those current in the theory of numbers are formally equivalent in the sense that either set may be inferred from the other. The forms in this paper are so stated that they provide powerful algorithms, in which all discussions of convergence are obviated by establishing identities between arithmetic functions. The theory can be generalized to associates $\varphi_{x, y, z, \ldots}$ of any number of independent classes (or abelian groups) of elements, but the generalization has no apparent interpretation in terms of ordinary numbers.

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