tain exactly 7 subgroups of order 9. Its order would therefore be 63, 126, or 252. This is impossible as each of these groups would involve only one subgroup of order 7 since each of its subgroups of order 7 would be transformed into itself by at least 21 substitutions.

It remains only to consider the case when $G_1$ would contain a substitution of order 3 and of degree 60 without involving such a substitution of degree 63. The order of the group formed by all the substitutions of $G$ which would be commutative with this substitution of order 3 would be 90. This group of order 90 would transform its ten subgroups of order 9 according to a transitive group of order 30 and of degree 10. Since this transitive group does not exist,* we have arrived at nothing but contradictions by assuming the existence of a second simple group of order $7!/2$ and hence such a group is actually proved to be non-existent.

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# A THEOREM OF OSCILLATION

BY W. E. MILNE

In an investigation of the oscillations of aerial bombs a need was found for the following proposition. Both the theorem and its proof are modelled after a similar theorem and proof by Osgood.†

THEOREM. *Let $\varphi(t)$ be positive, continuous, monotonically increasing, and bounded in the interval $T \leqq t < \infty$, and let $m$ and $M$ be two positive constants such that $m < \varphi(t) < M$ for $t > T$. Let $f(y)$ be an odd, monotonically increasing function, satisfying the Lipschitz condition*

$$|f(y_1) - f(y_2)| < K|y_1 - y_2|, \qquad K > 0,$$

*in an interval $-a \leqq y \leqq +a$, $a > 0$. Let $y$ be a solution of the differential equation*

$$\frac{d^2y}{dt^2} + \varphi(t)f(y) = 0 \tag{1}$$

* Cf. F. N. Cole, QUARTERLY JOURNAL, vol. 27 (1895), p. 40.

† This BULLETIN, vol. 25 (1919), pp. 216–221.

*subject to the conditions*

$$(2) \quad \frac{dy}{dt} = 0,\ y = y_1,\ |y_1| < a,\ f(y_1) \neq 0,\ \textit{when}\ t = t_1 > T.$$

*Then $y$ oscillates an infinite number of times in the interval $t_1 < t < +\infty$ and the amplitudes decrease monotonically but do not approach zero.*

Proof: Let us extend* the definition of $f(y)$ by the formulas

$$f(y) = f(a), \qquad \text{when} \quad y > a,$$
$$f(y) = f(-a), \qquad \text{when} \quad y < -a.$$

The function so extended satisfies the Lipschitz condition.

With the hypotheses thus extended, there exists† a unique function $y(t)$, continuous together with its first two derivatives, which satisfies (1) and (2) in the interval $t_1 \leqq t < \infty$. Now consider the case in which $y_1$ is positive. Then, at $t_1$, $d^2y/dt^2$ is negative and remains negative as long as $y$ is positive. Since

$$v = \frac{dy}{dt} = \int_{t_1}^{t} \frac{d^2y}{dt^2}\,dt, \qquad t > t_1,$$

we see that $v$ is negative as long as $y$ is positive. Therefore the graph of $y(t)$ as a function of $t$ is concave downward with negative slope to the right of $t_1$, and therefore must cut the axis at a finite point $t_1' > t_1$. Let $v_1$ be the corresponding value of $v$. Now multiply (1) by $2dy$ and integrate, obtaining

$$(3) \qquad v^2 = -2\int_{y_1}^{y} \varphi(t) f(y)\,dy.$$

At $t_1'$ this becomes

$$v_1^2 = 2\int_0^{y_1} \varphi(t) f(y)\,dy.$$

Since in the interval $t_1 \leqq t \leqq t_1'$ we have by hypothesis $\varphi(t_1) \leqq \varphi(t) \leqq \varphi(t_1')$, it follows that

$$v_1^2 \leqq 2\varphi(t_1')\int_0^{y_1} f(y)\,dy,$$

$$v_1^2 \geqq 2\varphi(t_1)\int_0^{y_1} f(y)\,dy.$$

Now let

$$\int_0^{y} f(y)\,dy = F(y).$$

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* This extension is made for convenience in establishing the existence of the solution. Actually the definition of $f(y)$ outside the interval $-a \leqq y \leqq a$ is immaterial.

† Bliss, PRINCETON COLLOQUIUM, p. 93.

Then $F(y)$ is even and continuous for $|y| < a$, is monotonically increasing in the interval $0 < y < a$, and vanishes at the origin. With this notation the above inequalities become

$$(4) \qquad \begin{cases} v_1^2 \leqq 2\varphi(t_1')F(y_1), \\ v_1^2 \geqq 2\varphi(t_1)F(y_1). \end{cases}$$

At $t_1'$ $v$ is negative; hence, immediately to the right of $t_1'$ $y$ is negative, and therefore $d^2y/dt^2$ is positive. Moreover as long as $v$ is negative, $d^2y/dt^2$ is monotonically increasing, as equation (1) shows. Then since

$$v = v_1 + \int_{t_1'}^{t} \frac{d^2y}{dt^2}dt,$$

it is clear that $v$ must vanish for a finite value of $t$, $t = t_2 > t_1'$. Let the corresponding value of $y$ be $y_2$. Now from (3)

$$v_1^2 = 2\int_0^{y_2}\varphi(t)f(y)dy,$$

whence as in the preceding case

$$(5) \qquad \begin{cases} v_1^2 \leqq 2\varphi(t_2)F(y_2), \\ v_1^2 \geqq 2\varphi(t_1')F(y_2). \end{cases}$$

From (4) and (5) we get

$$F(y_1) \geqq F(y_2), \qquad \text{or} \qquad |y_1| \geqq |y_2|,$$
$$\varphi(t_1)F(y_1) \leqq \varphi(t_2)F(y_2).$$

A similar argument leads to the same results when $y_1$ is negative.

Starting now with the conditions $dy/dt = 0$, $y = y_2$, when $t = t_2$, we may repeat the entire argument and obtain $|y_2| \geqq |y_3|$, $\varphi(t_2)F(y_2) \leqq \varphi(t_3)F(y_3)$, and in general

$$(6) \qquad |y_n| \geqq |y_{n+1}|,$$

and

$$(7) \qquad \varphi(t_n)F(y_n) \leqq \varphi(t_{n+1})F(y_{n+1}),$$

where $t_n$ is the $n$th value of $t$ (beginning with $t_1$) for which $dy/dt = 0$, and $y_n$ is the corresponding value of $y$.

The quantities $y_n$ are the amplitudes of the successive oscillations. Hence (6) proves that the amplitudes decrease monotonically. From (7), together with the hypotheses regarding $\varphi(t)$, it may be shown that $F(y_n) \geqq (m/M)F(y_1)$, which proves that the amplitudes do not approach zero.

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