THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and eighth regular meeting of the Society was held at Columbia University on Saturday, February 28, extending through the usual morning and afternoon sessions. The attendance included the following twenty-eight members:

Dr. J. W. Alexander, Professor R. C. Archibald, Professor A. A. Bennett, Professor Pierre Boutroux, Professor B. H. Camp, Professor F. N. Cole, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. H. Gronwall, Professor C. C. Grove, Professor Edward Kasner, Dr. E. A. T. Kircher, Mr. Harry Langman, President E. O. Lovett, Professor H. H. Mitchell, Mr. George Paaswell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. J. M. Stetson, Professor Oswald Veblen, Mr. H. E. Webb, Professor J. H. M. Wedderburn, Professor J. K. Whittemore.

Vice-President R. G. D. Richardson occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. F. J. Burkett, Pennsylvania State College; Mr. A. D. Campbell, Cornell University; Dr. Y. R. Chao, Cornell University; Professor R. E. Gilman, Brown University; Mr. D. C. Kazarinoff, University of Michigan; Dr. Norman Miller, Queen's University; Dr. G. M. Robison, Cornell University; Professor Jung Sun, Peking Academy; Dr. W. H. Wilson, State University of Iowa; Dr. S. D. Zeldin, Massachusetts Institute of Technology. Six applications for membership in the Society were received.

Professor Oswald Veblen was appointed to fill the unexpired term of Professor E. W. Brown, resigned, as representative of the Society in the Division of Physics of the National Research Council. The other representatives are Professor L. E. Dickson, reporter, and Professor H. S. White. These three representatives, the President and the Secretary of the Society, and Professors J. L. Coolidge and L. P. Eisenhart form a committee to consider the formation of an American Section of an International Mathematical Union. Professor Dickson is chairman of this committee.

Steps were taken to submit the question of the incorporation of the Society to the members at the April meeting.

The following papers were read at the February meeting:

(1) Professor JOSEPH LIPKA: "On the general problem of dynamics."

(2) Dr. A. R. SCHWEITZER: "On the iterative properties of the algebra of logic."

(3) Dr. A. R. SCHWEITZER: "On improper pseudogroups, with application to the abstract field."

(4) Mr. G. H. HARDY: "On the representation of numbers as sums of squares and in particular of five and seven."

(5) Dr. J. W. ALEXANDER: "On the representation of any n-dimensional two-sided manifold as a generalized Riemann surface."

(6) Dr. J. W. ALEXANDER: "On the equilibrium of a fluid mass at rest."

(7) Dr. T. H. GRONWALL: "Qualitative properties of the ballistic trajectory (second paper)."

(8) Dr. T. H. GRONWALL: "On the distortion in conformal mapping."

(9) Professor A. A. BENNETT: "Fictitious matric roots of the characteristic equation."

(10) Professor PIERRE BOUTROUX: "On multiform functions defined by differential equations of the first order."

(11) Professor B. H. CAMP: "The significance of a difference, and the value of a sample."

(12) Professor J. H. M. WEDDERBURN: "On division algebras."

(13) Professor EDWARD KASNER: "Geodesics of surfaces and higher manifolds."

Professor Boutroux's paper was a special expository presentation prepared at the suggestion of the programme committee. In the absence of the authors the papers of Professor Lipka, Dr. Schweitzer, Mr. Hardy, and Dr. Alexander's first paper were read by title. Abstracts of the papers follow below.

1. At the annual meeting of the Society, Professor Lipka presented the theorem: if a system of $\infty^{2(n-1)}$ trajectories in space of n dimensions is such that any ∞^{n-1} trajectories of the system which meet an arbitrary hypersurface (space of $n - 1$ dimensions) orthogonally admit of ∞^1 orthogonal hypersurfaces, then the system may be considered as the

trajectories in a conservative field of force with a given constant of energy. The purely geometric part of the theorem is true for all natural families of curves. The n-dimensional spaces considered were spaces of constant curvature. In the present paper, the above theorem is demonstrated for spaces of variable curvature. The result is the converse of the Lipschitz theorem announced in *Crelle's Journal*, volume 74 (1871).

2. Given a set S consisting of at least two elements, Dr. Schweitzer shows that this set constitutes an algebra of logic provided the following postulates are satisfied in addition to the property of unique closure with regard to the undefined relations $\theta(x, y)$ between the elements x, y of S:

1. $\theta(x, x) = \theta(y, y)$. 2. $\theta\{x, \theta(y, x)\} = x$.

3. There exists an element 0 such that for every x, y, z, $\theta\{x, \theta(y, z)\} = \psi\theta\{\psi\theta(x, y), \theta[z, \psi(x)]\}$, where $\psi(t) = \theta[0, t]$.

In the second part of the paper iterative relations based on the preceding postulates are given; these relations are mainly in generalization of the relations

$$\theta\{\theta(x, y), \theta(y, z)\} = \theta(x, y),$$

$$\theta\{\theta(y, x), \theta(z, y)\} = \theta(y, x).$$

Concretely, $\theta(x, y)$ may be interpreted by $x +$ not-y.

3. A formal (material) pseudogroup is any system of elements E satisfying the properties $S[\rho_1, \rho_2, \cdots]$ necessary for a formal (material) group and including closure with reference to the undefined relations ρ_i generating S, apart from a set E_0 of exceptional elements. If the set E_0 is empty, then the pseudogroup is proper; otherwise, improper. Dr. Schweitzer proves the following theorem: The abstract field is an improper formal pseudogroup $S[\phi, \psi, \phi_1, \psi_1]$ subject to the condition $\psi_1[x, \phi_1(y, x)] = y$. In case of the abstract group the undefined relations have the interpretations

$$\phi(x, y) = x \cdot y, \qquad \phi_1(x, y) = x^{-1}yx,$$

$$\psi(x, y) = x^{-1} \cdot y, \qquad \psi_1(x, y) = xyx^{-1},$$

and in case of the abstract field the relations may be interpreted thus:

$$\phi(x, y) = x + y, \qquad \phi_1(x, y) = x \cdot y,$$

$$\psi(x, y) = -x + y, \qquad \psi_1(x, y) = \frac{1}{x} \cdot y.$$

This result is analogous to a result obtained previously by the author for the algebra of logic.

4. Mr. Hardy's paper contains a detailed development of certain researches of which a short account has been published in volume 4, pages 189–193, of the *Proceedings of the National Academy of Sciences*. The paper will appear in the *Transactions* of the Society.

5. Dr. Alexander proves that any two-sided manifold of n-dimensions may be represented as a generalized Riemann surface spread over a hypersphere.

6. The question was raised by Liapounoff and Poincaré as to whether there exist any figures of equilibrium besides the sphere for a homogeneous fluid mass. In this paper, Dr. Alexander shows that there are no such figures whether of stable or unstable equilibrium.

7. The differential equations of the trajectory being

$$x'' = -G(v)H(y)x', \qquad y'' = -G(v)H(y)y' - g,$$

Dr. Gronwall shows that under fairly broad assumptions on the mode of increase of the functions G and H, the velocity v has only a finite number of maxima and minima as the time t varies from zero to infinity.

8. In the *Comptes rendus de l'Académie des Sciences de Paris*, February 28, 1916, Dr. Gronwall investigated the upper and lower bounds of $|w|$ and $|dw/dr|$, where

$$w = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

maps the circle $|z| < 1$ conformally on a simply connected and nowhere overlapping region in the w-plane, and the coefficient $a_2 = ae^{\gamma i}$ ($a > 0$, γ real) is given a priori. He obtained the exact values of these bounds, except the upper bound of $|z|$ in the case where $0 \leqq a < 1$, this case being inaccessible to the method used.

In the present paper, it is shown that for any r between zero and unity, and for $0 \leqq a < 1$, we have for $|z| = r$

$$|w| < \frac{1}{4}\left\{ \frac{1}{1-a}\left[1 - \left(\frac{1-r}{1+r}\right)^{1-a}\right] - \frac{1}{1+a}\left[1 - \left(\frac{1-r}{1+r}\right)^{-1-a}\right]\right\},$$

except for the function w obtained by replacing r by $ze^{-\gamma i}$ in the expression to the right, the upper bound being then attained for $z = re^{\gamma i}$.

9. In this paper Professor Bennett shows that many of the simple relations connected with the characteristic equation of a square matrix of order n become obvious if there be assumed to exist n formal matrix quantities (one of which is the given matrix) which are roots of the characteristic equation, commutative with the given matrix, and for which the elementary symmetric functions are scalars. These formal matrices do not always have an actual existence, and their use corresponds in a sense to the introduction of imaginary numbers into algebra.

10. Professor Boutroux's paper first gives a classification of the different types of differential equations belonging to the family $y' + A_0 + A_1y^2 + A_3y^3 = 0$, where the A's are polynomials in x. It then solves, for the simplest type, a problem which has been briefly characterized in the abstract of a paper presented (by title) at the Saint Louis meeting of the Society (see March BULLETIN, pages 271–272).

11. The usual criterion of Tchebycheff for evaluating the probability of the existence of a datum further from the mean value of a distribution than a given multiple of the standard deviation is known to be inadequate. Closer tests are derived by Professor Camp, and applied to the problems of finding the significance of a difference and the value of the mean of a sample. New inequalities used in connection with the point binomial permit a rigorous demonstration of the theorem that, when the distribution from which the

samples are drawn is a point binomial, the distribution of the means may be considered to be the Gaussian law without appreciable error. The point binomial is found to have an invariant property similar to that enjoyed by the Gaussian law.

12. In this paper, Professor Wedderburn shows that the only non-commutative division algebras of order 9 are of the type discovered by L. E. Dickson.

13. Professor Kasner discusses first the ∞^1 geodesics of a surface passing through a given point, giving theorems about the distribution of the osculating spheres. He then studies the ∞^1 plane curves obtained by projecting the geodesics on the tangent plane. These curves all have zero curvature; but J, the rate of variation of curvature with respect to arc, is shown to be a cubic function of the slope. The reconstruction of the surface from the ∞^1 plane curves is then discussed.

For a curved three-dimensional manifold we have ∞^2 geodesics at a point. These are projected on the tangent three-flat, thus giving a bundle of curves in ordinary space. It is shown that these curves have inflexions, the torsion vanishes, but a cubic law is obtained for J. To each tangent line there corresponds an osculating plane; this gives rise to a certain integration problem, and integral cones of the form $X^{\alpha}Y^{\beta}Z^{\gamma} =$ constant are produced, which are recognized as W-cones (analogous to W or anharmonic curves). The correspondence between tangent and binormal for the curves of the bundle is found to be a quadratic Cremona transformation, and is simply visualized and determined by the indicatric quadric surface of the given manifold at the given point. Generalizations for four-dimensional manifolds (required in Einstein's theory) are stated.

Finally the author extends to general manifolds his theorems concerning the limit of the arc to the chord given in this BULLETIN, volume 20 (1914), pages 524–531. This is of importance because when the minimal lines are real, as in the Einstein theory of gravitation, where they appear as light rays, the peculiar limits obtained instead of unity, namely 0.94, 0.86, 0.80, etc., have physical significance.

F. N. COLE,
Secre ary.