THE TWENTY-FOURTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

At the invitation of Adelbert College and the Case School of Applied Science, Cleveland, Ohio, the twenty-fourth summer meeting of the Society was held at these institutions on Tuesday, Wednesday, and Thursday, September 4–6, 1917. This was the Society's second visit to Cleveland, the annual meeting having been held there in the winter of 1912–1913. On the present occasion the interest was reinforced by the meeting of the Mathematical Association of America, immediately following on September 6–7. The arrangements, which were in charge of a committee representing both organizations, included a joint session on Thursday morning, at which Professor L. P. Eisenhart presented an address on "Darboux's Contribution to Geometry," and a joint dinner on Wednesday evening, attended by seventy-six members and friends, to whom President Thwing, of Western Reserve University, spoke a word of greeting, which was followed by a number of informal responses to the calls of the toast-master, Professor Huntington. The programme on Wednesday afternoon included an inspection of the harmonic analysis apparatus of Professor Miller, of the Case School, and an organ recital in the chapel. On Thursday afternoon President Thwing gave a garden party in honor of the visiting societies. Luncheon was served on each day at the Case Club, whose building was thrown open to the members afternoons and evenings. At the close of the meeting a vote of thanks was tendered to the authorities of the two colleges for their generous hospitality.

The meeting included the usual morning and afternoon sessions on Tuesday and Wednesday and the joint session on Thursday morning. The following sixty-four members were in attendance:

Professor O. P. Akers, Dr. Florence E. Allen, Professor R. B. Allen, Professor L. D. Ames, Professor Frederick Anderegg, Professor R. C. Archibald, Professor G. N. Armstrong, Professor Grace M. Bareis, Professor J. W. Bradshaw, Dr. R. W. Burgess, Professor W. D. Cairns, Professor Florian Cajori, Professor W. M. Carruth, Professor W. B. Carver, Mr. E. H.

Clarke, Dr. G. R. Clements, Professor F. N. Cole, Professor A. R. Crathorne, Professor C. H. Currier, Professor F. F. Decker, Professor L. L. Dines, Professor L. W. Dowling, Professor John Eiesland, Professor L. P. Eisenhart, Professor T. M. Focke, Professor Tomlinson Fort, Dr. M. G. Gaba, Professor D. C. Gillespie, Professor O. E. Glenn, Professor W. C. Graustein, Professor C. F. Gummer, Professor A. M. Harding, Professor H. E. Hawkes, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor F. H. Hodge, Professor E. V. Huntington, Professor W. A. Hurwitz, Dr. R. A. Johnson, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor H. G. Keppel, Professor G. A. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor F. R. Moulton, Professor A. D. Pitcher, Professor L. C. Plant, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor Maria M. Roberts, Professor E. D. Roe, Jr., Professor D. A. Rothrock, Professor J. E. Rowe, Professor C. H. Sisam, Professor P. F. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor E. B. Stouffer, Professor M. O. Tripp, Professor D. T. Wilson, Professor B. F. Yanney, Professor J. W. Young.

At the opening session on Tuesday morning Professor Focke occupied the chair, which was filled in succession by Professors Hedrick, Cajori, G. A. Miller, and Eisenhart. Professor Hedrick presided at the joint session. The Council announced the election of the following persons to membership in the Society: Dr. W. L. Crum, Yale University; Professor T. J. Fitzpatrick, University of Nebraska; Dr. T. R. Hollcroft, Columbia University; E. L. Ince, M. A., Trinity College, Cambridge, England; Mr. L. S. Odell, Manual Training High School, Brooklyn, N. Y.; Dr. T. A. Pierce, Harvard University. Five applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Professor ARNOLD EMCH: "On the invariant net of cubics in the Steinerian transformation."

(2) Professor J. E. ROWE: "Theorems related to a point projection of the rational plane cubic curve."

(3) Professor J. E. ROWE: "Closed hexagons related to the rational plane cubic curve."

(4) Professor J. E. ROWE: "The projections of certain points upon the rational plane quartic curve."

(5) Professor TOMLINSON FORT: "Some theorems of comparison and oscillation."

(6) Professor O. D. KELLOGG: "Oscillation and interpolation properties of solutions of integral equations."

(7) Professor A. B. COBLE: "Finite groups determined by $2p + 2$ points in S_p."

(8) Dr. M. G GABA: "Complete existential theory of the postulates of the linear order η."

(9) Professor L. L. DINES: "The bordered Fredholm determinant and the related group of functional transformations."

(10) Professor R. G. D. RICHARDSON: "Contributions to the study of oscillation properties of ordinary linear differential equations of the second order."

(11) Professor C. N. MOORE: "On the summability of the developments in Bessel's functions."

(12) Professor G. A. MILLER: "Groups formed by special matrices."

(13) Professors VIRGIL SNYDER and F. R. SHARPE: "On the space involution of order 8 defined by a web of quadric surfaces."

(14) Dr. R. W. BURGESS: "A second approximation for cantilevers."

(15) Professor FLORIAN CAJORI: "L. Wantzel."

(16) Dr. G. M. GREEN: "Conjugate nets with equal point invariants."

(17) Dr. G. M. GREEN: "Plane nets with equal invariants."

(18) Professor FLORIAN CAJORI: "Newton's solution of numerical equations by the use of slide rules."

(19) Professor L. P. EISENHART: "Transformations of planar nets with equal invariants."

(20) Dr. L. C. MATHEWSON: "On the group of isomorphisms of a certain extension of an abelian group."

(21) Professor E. D. ROE, JR.: "Some restricted developments."

(22) Professor E. D. ROE, JR.: "A geometric representation. Second paper."

(23) Professor E. D. ROE, JR.: "Integral functions as products."

(24) Mrs. E. D. ROE, JR.: "Interfunctional expressibility problems of symmetric functions. Second paper."

(25) Professor E. L. DODD: "The approximation or graduation of a mortality table by means of a sum of exponential functions."

(26) Professor D. C. GILLESPIE: "Repeated integrals."

(27) Professor W. A. HURWITZ: "An expansion theorem for systems of linear differential equations."

(28) Professor W. C. GRAUSTEIN: "Note on isogenous complex functions of curves."

(29) Dr. MARY F. CURTIS: "A proof of the existence of the functions of the elliptic cylinder."

(30) Professor JOHN EIESLAND: "A Plücker geometry of flats in odd n-space."

(31) Professor H. J. ETTLINGER: "Theorems of oscillation for a generalized Sturmian boundary problem."

(32) Professor H. J. ETTLINGER: "Theorems of oscillation for the general real, self-adjoint system of the second order."

(33) Professor E. V. HUNTINGTON: "Bibliographical note on the use of the word mass in current textbooks."

(34) Professor L. P. EISENHART: "Darboux's contribution to geometry."

Dr. Curtis was introduced by Professor Bôcher. Dr. Green's second paper was presented by Professor Eisenhart, and Mrs. Roe's paper by Professor Roe. The papers of Professor Emch, Professor Coble, Dr. Mathewson, Professor Dodd, Professor Eiesland, and Professor Ettlinger, and Dr. Green's first paper were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond with the titles in the list above.

1. Considering $A_1(1, 0, 0)$; $A_2(0, 1, 0)$; $A_3(0, 0, 1)$; $E(1, 1, 1)$ as the fundamental triangle and unit point of a system of projective coordinates, the formulas for the Steinerian transformation associated with the quadrangle $A_1A_2A_3E$ are

$$\rho x_1' = x_1(x_2 + x_3 - x_1),$$
$$\rho x_2' = x_2(x_3 + x_1 - x_2),$$
$$\rho x_3' = x_3(x_1 + x_2 - x_3).$$

The net of cubics invariant under this transformation is represented by

$$a_1x_2x_3(x_2 - x_3) + a_2x_3x_1(x_3 - x_1) + a_3x_1x_2(x_1 - x_2) = 0.$$

Every triad of values $a_1 : a_2 : a_3$ defines a cubic of the net through the quadrangle and its diagonal points. Every cubic has the interesting property that the tangents at A_1, A_2, A_3, E meet in a point of the cubic with the coordinates a_1, a_2, a_3. Professor Emch discusses some geometric properties of this net of cubics.

2. In Professor Rowe's first paper four theorems are established, the first two of which are summarized in the statement: If $\varphi_{t_1}{}^2\varphi_{t_2}{}^2 = 0$ is the equation of the line determined by the two points t_1, t_2 of the rational plane cubic curve R^3, the discriminant of this quadratic in t_2, equated to zero, yields a quartic in t_1 whose roots are the parameters of the residual points cut out of the R^3 by the four tangents to the R^3 from the point x_i ($i = 0, 1, 2$) of the plane; or for a fixed value of t_1 this same equation is the equation of a tritangent conic of the R^3 whose points of contact are t_1 and the points of contact of the two tangents that may be drawn from t_1 to the R^3. Theorems III and IV follow.

III. The parameters of the points of contact of the two tangents to the R^3 from any point of itself are harmonic to the parameters of the node.

IV. If t_2, t_3 are the parameters of the points of contact of the two tangents to the R^3 from the point t_1, and t_1' is the third point of the R^3 collinear with t_2, t_3, the transformation which sends t_1 into t_1' is involutory.

3. In this paper Professor Rowe derives one form of the necessary and sufficient condition that six points of the rational plane curve R^3 of order three lie on a conic, and uses this condition to prove the following theorems:

I. If six points of the R^3 lie on a conic, the tangents to the R^3 at these six points cut out of the R^3 six points on a conic.

II. If six points of the R^3 lie on a conic, the sides of the closed hexagon formed with these points as vertices cut the R^3 in six points on a conic.

III. If from a point of the R^3 the two tangents are drawn to the R^3 and from each point of contact of these two tangents the other two tangents to the R^3 are drawn, the six points of contact of the six tangents lie on a conic.

By a limit process it is evident that Theorem I may be derived as a corollary of Theorem II, although a different proof is given for each.

Also, it may be verified that the six points of contact of the three pairs of tangents drawn to the R^3 from three of its collinear points do not, in general, lie on a conic.

4. In this paper Professor Rowe proves the following four theorems together with three corollaries:

I. The parameters of the points of contact of the two tangents that can be drawn to the rational plane quartic curve R^4 from one of its nodes are harmonic to the parameters of the residual points cut out of the R^4 by any line through this node.

II, III, IV are summarized in the statement: If t_1, t_2 are the parameters of the points of contact of two flex tangents, of two double tangents, or of a flex tangent and a double tangent of the R^4, which cut out of the R^4 the residual points t_1', t_2', and if t_1'', t_2'' are the parameters of the points of contact of the other two tangents that can be drawn to the R^4 from the intersection of the tangents at t_1, t_2, the pairs (t_1, t_2), (t_1', t_2'), (t_1'', t_2'') form a set in involution.

5. Let y denote a solution of the differential equation $d(k(x)y')/dx + G(x)y = 0$. Professor Fort develops a number of theorems relative to $\alpha_1 k(x)y' + \alpha_2 y \equiv V(x)$ which reduce to well known theorems of comparison, oscillation, etc., in case that $V(x) \equiv y$.

6. In the *American Journal of Mathematics* for January, 1916, Professor Kellogg showed that continuity and orthogonality of a function set on the interval (0, 1) was insufficient to insure oscillation, but that if to these assumptions is added the hypothesis (D): the determinants $|\varphi_i(x_j)|$ are positive for $0 < x_0 < x_1 < x_2 \cdots < 1$, a considerable number of the properties of the more common sets of orthogonal functions follow. The present paper derives a sufficient condition on the kernel of an integral equation in order that its characteristic solutions may have the property (D) and its consequences.

7. It is the purpose of Professor Coble's paper (to appear in the *American Journal of Mathematics*) to show that a set

of $2p + 2$ points in a projective space S_p, which is such that the quadrics on $2p + 1$ of the points will pass through the remaining one, will define by purely projective processes a finite group which is isomorphic with the group determined by the odd and even theta functions in p variables. Since both the number of absolute constants of the point set and the number of moduli of the general theta function is $\frac{1}{2}p(p + 1)$, this indicates an algebraic connection between such point sets and the general theta modular functions. This connection has been set forth hitherto only in the particular cases, $p = 1, 2, 3$, and in the hyperelliptic case for all values of p; although in an earlier paper of the writer a grouping of the theta characteristics isomorphic with a grouping of the points of the set has been exhibited. It appears thereby that the point set may serve as the basis for a geometrical classification of the general theta functions, just as the algebraic curve of genus p has served for the discussion of the Riemannian theta functions of genus p.

8. In the March, 1917, BULLETIN, Professor E. V. Huntington gave three sets of completely independent postulates for serial order. His set A involved four postulates, which is as high a number as has been proved to be completely independent. In the present paper Dr. Gaba states seven postulates for the linear order η which form a categorical and completely independent set. For his 128 independence examples he gives 16 sets of points and 8 definitions of $<$ such that each definition is applicable to every one of the 16 sets of points and each combination of set and definition of $<$ yields a different example.

9. The determinant of the projective transformation

$$\phi'(x) = \frac{\alpha(x) + \beta(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy}{\delta + \int_0^1 \epsilon(y)\phi(y)dy},$$

is defined by Professor Dines to be the bordered Fredholm determinant

$$B = D\left[\delta - \int_0^1 \epsilon(y)\bar{\alpha}(y)dy\right] + \int_0^1\int_0^1 \epsilon(x)D_1(x, y)\bar{\alpha}(y)dxdy,$$

where $\bar{\alpha}(x) = \alpha(x)/\beta(x)$, and D and $D_1(x, y)$ are respectively the Fredholm determinant and first minor of the kernel $\gamma(x, y)/\beta(x)$. The following results are obtained:

(1) The determinant of the product of two projective transformations is equal to the product of their determinants.

(2) If the determinant B of a projective transformation is different from zero, a unique inverse transformation exists and is expressible in terms of B and four suitably defined first minors of B.

(3) The group of non-singular projective transformations (those for which $B \neq 0$) contains every finite transformation generated by a regular infinitesimal projective transformation as defined by Kowalewski; and, conversely, any non-singular projective transformation which does not differ too greatly from the identity transformation can be generated by such an infinitesimal transformation.

10. The Sturmian problem of determining the existence of parameter values λ for which the equation

$$(py_x)_x + G(x, \lambda)y = 0, \qquad p(x) > 0,$$

has solutions $y(x)$ satisfying linear self-adjoint boundary conditions has been much studied on account of its importance in mathematical physics and because it is a typical linear problem. Professor Richardson announces the results of further investigations as follows: (1) the derivation of sufficient conditions to be imposed on the function $G(x, \lambda)$ to insure the existence of solutions oscillating as many times as desired; (2) the determination of conditions on $G(x, \lambda)$ such that for a given interval of λ there is precisely one or precisely two parameter values for which exist solutions satisfying the boundary conditions and oscillating a prescribed number of times; (3) the completion of the theory of oscillation for the most important special case $G(x, \lambda) \equiv q(x) + \lambda k(x)$, viz., an investigation of the only case not hitherto studied in detail. For the non-definite case ($k(x)$ both signs, $q(x)$ positive in at least a part of the x-interval) it is found that there exist two integers ν_1, ν_2 ($\nu_2 \geqq \nu_1 \geqq 0$) such that if n denote the number of oscillations of the solution then for $n < \nu_1$ there are no solutions, for $n > \nu_1$ there are at least two, and for $n \geqq \nu_2$ there are precisely two.

11. The principal theorems obtained in Professor Moore's paper are the following: (A) If $f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, approaches a finite limit as x approaches $+0$, and furthermore is such that $(f(x) - f(+0))/x^\delta$, where $\delta > \frac{1}{2}$, remains finite in the neighborhood of the origin, then the development of $f(x)$ in Bessel's functions of order $\nu (\nu \geqq 0)$ will be summable $(C, k > \frac{1}{2})$ to the value $f(+0)$ for $x = 0$, provided $\nu = 0$ or $f(+0) = 0$, and will be uniformly summable $(C, k > \frac{1}{2})$ to the value $f(x)$ in any interval $0 \leqq x \leqq c \leqq 1$, throughout which $f(x)$ is continuous, provided $\nu = 0$ or $f(+0) = 0$. (B) If $\sqrt{x}f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, then the development of $f(x)$ in terms of Bessel's functions of order $\nu (\nu \geqq 0)$ will be summable $(C, k > 0)$ to the value $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at every point in the interval $0 < x \leqq 1$ at which $f(x)$ is continuous or has a finite jump, and will be uniformly summable $(C, k > 0)$ to $f(x)$ throughout any closed interval lying entirely in the interval $0 < x \leqq 1$ and throughout which $f(x)$ is continuous.

12. Any set of square matrices of order n having one and only one unity element in each row and in each column, while all their other elements are zeros, generates a finite group which is simply isomorphic with some substitution group of degree n. When these unity elements are replaced by kth roots of unity the resulting matrices generate a group which is simply isomorphic with a substitution group of degree kn. When this substitution group is transitive it has n systems of imprimitivity, each system being of degree k. The main object of Professor Miller's paper is to consider the nature of these elements in order that any imprimitive group of degree kn having n systems of imprimitivity can be represented by such square matrices of order n. The following theorems are established: Every possible imprimitive group of degree $2n$ which has n systems of imprimitivity is simply isomorphic with a group formed by square matrices of order n having one and only one ± 1 element in each row and in each column while all their other elements are zeros. If an imprimitive group of degree kn has for its head H the direct product of n regular cyclic groups of order k, then every invariant imprimitive subgroup has for its head a subgroup of H which is simply isomorphic with the direct product of $n - 1$ cyclic groups of order k and of some subgroup of such a cyclic group.

13. The involution discussed by Professors Snyder and Sharpe is defined by associating the planes of one space with the quadric surfaces of a web in another. We have a (1, 8) correspondence between the points of the two spaces, and an involution of order 8 in the space of the quadrics. The surface of coincident points, the surface of branch points, and the residual surface are obtained and described. The dual of the surface of branch points is the symmetroid, the characteristic properties of which are easily obtained by this method. As particular cases we meet with the Weddle surface, the Kummer surface, the Hessian of the cubic, and all of the line congruences of order 2.

14. The engineer uses constantly certain simple formulas for the deflection of a horizontal beam fixed at one end and bent by a vertical force at the other, and for the angle of slope of the free end. The treatment by which these formulas are derived can not readily be modified to give true results for large deflections; but by using the equation for the real elastic curve and developing in series the elliptic integrals involved, Dr. Burgess obtains the following formulas for the deflection d, the horizontal component t of the displacement of the free end, and the angle of slope ψ:

$$d = \frac{l^3W}{3EI} - \frac{4}{105}\frac{l^7W^3}{E^3I^3} + .008\frac{l^{11}W^5}{E^5I^5}$$

$$t = \frac{1}{15}\frac{l^5W^2}{E^2I^2} - \frac{4}{315}\frac{l^9W^4}{E^4I^4}$$

$$\tan\psi = \frac{1}{2}\frac{l^2W}{EI} - \frac{1}{240}\frac{l^6W^3}{E^3I^3} + \frac{l^{10}W^5}{600E^5I^5},$$

where E is the constant of elasticity, W the deflecting force, l the length of the beam, and I the geometric moment of inertia of a cross-section about an axis perpendicular to the plane of bending. The first terms in the expressions for d and $\tan\psi$ are the engineers' formulas; the other terms give corrections important for larger deflections, and apparently sufficient within the elastic limit.

15. Professor Cajori gave some details on the life and work of Pierre-Laurent Wantzel (1814–1848) of the Polytechnic School in Paris.

16. In his first paper, Dr. Green gives a new geometric characterization of conjugate nets with equal point invariants, which is a refinement of a theorem established by him in a previous paper.* The following theorem is first proved geometrically. Let t_1 and t_2 be the ray tangents at a point of a surface S referred to a conjugate net. On the corresponding ray there are two focal points F_1 and F_2, corresponding respectively to the ray curves of which t_1 and t_2 are the tangents. Then the tangent to S which is conjugate to t_1 meets the ray in F_2, and the tangent conjugate to t_2 meets the ray in F_1.

It is then proved that a necessary and sufficient condition that a conjugate net have equal point invariants is that the ray tangents meet the ray in the focal points of the ray, each ray tangent passing through the focal point which does not correspond to it.

If the ray tangents meet the ray in the corresponding focal points of the ray, then the ray curves coincide with the asymptotic net, and the conjugate net has not equal invariants.

17. The geometric characterization of conjugate nets with equal point invariants given in Dr. Green's first paper does not admit of immediate extension to the case of planar nets with equal invariants. For any planar net N, the ray of a point may be defined as the line joining the minus first and first Laplace transforms of the point, but every one-parameter family of rays forms a developable and gives rise to a focal point on each ray. The first object of Dr. Green's second paper is to define a net N' congruentially associated with the net N. Let t_1, t_2 be the tangents to the curves of N at a point of the plane, and let t_1', t_2' be the tangents to the curves of N' at the same point. The curve of N' to which t_1' is tangent gives rise to a focal point F_1 on the ray of the net N. Similarly, a focal point F_2 on this ray corresponds to the tangent t_2'. The net N' is said to be congruentially associated with N if the harmonic conjugate of t_1' with respect to t_1, t_2 passes through F_2, and the harmonic conjugate of t_2' with respect to t_1, t_2 passes through F_1. It is proved that there always exists one and only one net congruentially associated with the given one.

A necessary and sufficient condition that the net N have

* *Amer. Jour. of Mathematics*, vol. 38 (1916), p. 313.

equal invariants is that at every point the tangents to the two curves of N separate harmonically the tangents to the two curves of the congruentially associated net N', or, in other words, that each ray of N be met by the tangents of N' in its focal points with respect to N'.

18. Professor Cajori notes three accounts of Newton's solution of numerical equations by the use of logarithmic slide rules, only one of which is given in the collected works of Newton and is referred to in the Encyclopédie des Sciences mathématiques, Tome I, Volume 4, page 429, namely, four sentences in a letter of Oldenburg to Leibniz of June 24, 1675. A second fuller account is described in E. Stone's New Mathematical Dictionary, second edition, London, 1743 and is reprinted in Cajori's History of the Slide Rule, 1909. A third account, purporting to be taken directly from Newtonian manuscripts, is printed in James Wilson's Mathematical Tracts of the late Benjamin Robins, London, 1761. The three accounts contain four slightly different modes of procedure.

19. In the January number of the *Transactions* Professor Eisenhart developed a theory of transformations of a conjugate system of curves on any surface into conjugate systems on other surfaces. These results can be extended so as to give transformations of conjugate nets on two-dimensional spreads in space of any order. In the present paper he applies them to nets in the plane, and in particular to nets with equal point invariants. In the June number of the *Annals of Mathematics* it was shown how by quadratures one can obtain from such a net a system of asymptotic curves on a surface which are perspective with the net from a point. Two nets with equal invariants in the relation of a transformation here discussed are found to be perspective with two surfaces which are the focal surfaces of a W-congruence. The planar nets of period three under transformations of Laplace have equal invariants. The paper deals also with transformations of these nets into nets of the same kind, and establishes the existence of a theorem of permutability of these transformations. The analysis involved in this investigation is capable also of interpretation as giving transformations of quadrics, certain ruled surfaces, and surfaces of the kind first discussed by Tzitzeica into surfaces of these respective types.

20. In 1908 Professor G. A. Miller showed that "if an abelian group H which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of H into its inverse, then the group of isomorphisms of this extended group is the holomorph of H."* The present paper by Dr. Mathewson discusses an elaboration of Professor Miller's theorem. In toto it is proved that if G is formed by extending an abelian group H which has operators of order > 2 by a certain operator which transforms every one of its operators into the same power of itself and which is commutative with no operator of odd order in H, then the group of isomorphisms of G is the holomorph of H, and is a complete group if H is of odd order.† In the proofs of the various theorems of the paper free use is made of properties of rational integers, which are either quoted or established.

21. In this paper in an elementary way without the assumption of Fourier's series, Professor Roe derives the two developments

$$f(x) = f(a) + \sum_{r=1}^{r=\infty} \int_a^x (-1)^{r-1} \frac{\sin (2rf'(x))}{r} dx,$$

where the range of a and x is determined by

$$\pi/2 \geqq f'(x) \geqq -\pi/2;$$

and

$$f(x) = f(a) + \frac{\pi}{2}(x-a) - \sum_{r=1}^{r=\infty} \int_a^x \frac{\sin (2rf'(x))}{r} dx,$$

where the range of a and x is determined by

$$\pi \geqq f'(x) \geqq 0.$$

In both cases the range is varied by suitable transformations. Numerous applications to the development of functions and the summations of certain series are given.

22. This paper is a continuation and further elaboration of matters suggested in Professor Roe's previous paper, presented at the October, 1916, meeting.

* Miller, *Philosophical Magazine*, vol. 231 (1908), p. 224.

† Miller, *Mathematische Annalen*, vol. 66 (1908), p. 135.

The interpretation of the complex values of y in $y = f(x, z)$, surface, is given, and it is shown how the equations of surfaces may be expressed by the method considered, and how the equations of families of spirals in two coordinates on surfaces are derived. The spiral systems arising from the complex values of y in $y = \{x^{x-1}(x - 1)^x\}^{1/(1-2x)}$ together with the value systems of this function are considered in detail. Special illustrations are given.

23. Professor Roe gives in this paper from an elementary standpoint the necessary and sufficient conditions for developing integral functions into products, and thus furnishes a method for developing into products series whose zeros are known or can be found, the results of which agree with those found by the methods of the theory of functions. Illustrations are furnished.

24. In this paper Mrs. Roe continues the investigation begun in her paper presented at the October, 1916, meeting. From the product and quotient tables she derives new formulas for the coefficients of each of the complexes involved, in terms of each of the others, together with new proofs of symmetry in certain of the tables of such coefficients.

25. For the Makeham approximation or graduation the English H^m Mortality Table was divided at the age 28. Professor Dodd finds it convenient to divide the American Experience Table at the age 70, and to approximate each portion separately by means of a more simple function, viz.,

$$a + br^x + cs^x.$$

This function gives very closely the number of survivors at the age x. The terms before and after multiplication by the powers of the discounting factor give rise to geometric progressions, making easy the computation of the life functions—including increasing life functions—at any desired rate of interest. This assists in the problem of determining the interest rate which the insured actually realizes, on the basis of the gross premium. In constructing or graduating a table by the above function from raw material, the least square method may be employed readily, after first approximations have been obtained.

26. The theorem of the note by Professor Gillespie is: the repeated integral of the bounded function $f(x, y)$ may be taken in either order with equal results; i. e.,

$$\int_0^1 dx \int_0^1 f(x, y)dy \quad \text{and} \quad \int_0^1 dy \int_0^1 f(x, y)dx$$

are equal where they both exist.

27. Professor Hurwitz presents an analogue to the Sturm-Liouville theory, leading to the expansion of two given functions in terms of solutions of a pair of linear ordinary differential equations with linear parameter, satisfying certain linear boundary conditions, the coefficients in the two expansions being the same.

28. In this note Professor Graustein proves the following theorem in connection with Volterra's theory of isogenous complex functions of curves: If the gradients c_1, c_2 of the functions F_1, F_2 in $F = F_1 + iF_2$, where $F_1 = F_1|[L]|$, $F_2 = F_2|[L]|$ are functions of the first degree of the space curve L, are in general analytic, the gradients γ_1, γ_2 of Φ_1, Φ_2 in $\Phi = \Phi_1 + i\Phi_2$, an arbitrary complex function of L of the first degree isogenous with F, are analytic save perhaps in points of singularity of c_1 or c_2 and points in which both these vectors are indeterminate.

29. Heine's proof of the existence of the functions of the elliptic cylinder, that is, of the existence of real periodic solutions of period 2π of the differential equation

$$\frac{d^2E(\varphi)}{d\varphi^2} + 4\left(\frac{2}{b}\cos 2\varphi + z\right)E(\varphi) = 0, \qquad b \neq 0,$$

—a differential equation fundamental in physical problems dealing with elliptic cylinders and elliptic membranes—is clearly at fault in an essential point. Dannacker's attempt to remedy the error is unsatisfactory. In the present paper Dr. Curtis sets up the proof in a rigorous fashion.

30. In Professor Eiesland's paper an attempt has been made to develop a geometry of flats in odd n-space on the same lines as Plücker's geometry in 3-space. Rectangular coordi-

nates are used throughout and some interesting results are obtained. The subject has of necessity led to certain projective properties of n-space which are believed new. The flat-complex seems to contain far-reaching possibilities when completely developed. One thinks especially of the interpretation of flat-geometry as sphere-geometry on the one hand and on the other as anallagmatic geometry in the next higher space, such as has been treated by Professor Ranum.

31. Professor Ettlinger generalizes the Bôcher-Sturm oscillation theorem to the following system:

$$d[K(x, \lambda)u_x(x, \lambda)]/dx - G(x, \lambda)u(x, \lambda) = 0,$$

$$L[a, u(a, \lambda)] = 0, \quad L[b, u(b \ \lambda)] = 0,$$

where

$$L[x, u(x, \lambda)] = \alpha(x, \lambda)u(x, \lambda) - \beta(x, \lambda)K(x, \lambda)u_x(x, \lambda).$$

In general the same conditions are placed on the coefficients of the differential equation and the boundary conditions* except that $\alpha(a, \lambda)/\beta(a, \lambda)$ and $-\alpha(b, \lambda)/\beta(b, \lambda)$ decrease as λ increases, and if $\beta(x, \lambda) \not\equiv 0$ and

$$\{\alpha\beta\} = \alpha_x(x, \lambda)\beta(x, \lambda) - \alpha(x, \lambda)\beta_x(x, \lambda) + \alpha^2(x, \lambda)/K - \beta^2(x, \lambda)G,$$

then $\lim (\lambda \rightarrow \Lambda_1) \max \{\alpha\beta\} < 0$ and $\{\alpha\beta\}$ changes sign once and only once as λ increases. The existence of an infinite set of characteristic numbers for this system is proved and oscillation theorems for $L[x, u_p(x)]$ and $u_p(x)$ are established. The Sturm-Bôcher theorem appears as a special case of this result, where a particular form is assigned to $\alpha(x, \lambda)$ and $\beta(x, \lambda)$.

32. In a forthcoming paper in the *Transactions* Professor Ettlinger extends the work of Birkhoff concerning the solutions of a general self-adjoint linear system of the second order. The object of the present paper is to generalize these results to a system with more general boundary conditions, and to derive more exact oscillation theorems.

Consider the second order linear differential equation

* See Bôcher, Leçons sur les Méthodes de Sturm, Paris, 1917, p. 63.

$$d[K(x, \lambda)u_x(x, \lambda)]/dx - G(x, \lambda)u(x, \lambda) = 0,$$

$$L_i[a, u(a, \lambda)] = L_i[b, u(b, \lambda)] \qquad (i = 1, 2),$$

where

$$L_i[x, u(x, \lambda)] = \alpha_i(x, \lambda)u(x, \lambda) - \beta_i(x, \lambda)K(x, \lambda)u_x(x, \lambda).$$

Under the corresponding sets of conditions, which are the natural extensions of those imposed in the earlier paper, the existence of an infinite set of parameter values for this system is established and an exact oscillation theorem obtained for $L[x, u_p(x)]$ and a slightly less exact theorem for the oscillations of $u_p(x)$.

The case considered above yields the other (see the earlier paper) by assigning a special form to the coefficients of the boundary conditions.

The work is based on the generalized Sturmian theorem of oscillation (see the preceding abstract).

33. As a result of an examination of the definition of mass in a large number of current books on mechanics and physics, Professor Huntington finds the following principal ways in which the mass of a body is said to be measured: (1) by force per acceleration (inertia); (2) by the beam balance (standard weight); (3) by mutual acceleration (Mach's definition); (4) by the number of supposed identical particles in the body. The pseudo-definition of mass as "quantity of matter," and various circular definitions, are condemned. The conditions for measurability in general are stated, and suggestions in regard to the best use of the term mass are discussed. The paper will be offered to the *American Mathematical Monthly*.

34. Professor Eisenhart's address, delivered before a joint session of the Society and the Mathematical Association, was devoted to an analysis of Darboux's contributions to geometry. This address will be published in the January issue of the BULLETIN.

F. N. COLE,
Secretary.