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ON FOCI OF CONICS.

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THE study of the properties of conic sections, and of certain related points and lines has afforded ample scope for the faculties of all mathematicians from the Greeks on down through the ages. And not the least important among the points connected with the sections are the foci. It is the object of the present paper to give (I) a short historical sketch of the development of the properties of conics connected with the foci; (II) some of the theorems from Pappus which have a bearing on foci and tangents.

I. *Historical Sketch.*

What name should be connected with the discovery of the foci is still a matter of conjecture. Zeuthen* seems to think

* Geschichte der Mathematik im Alterthum und Mittelalter, Kopenhagen, 1896, p. 211.

that the focus for the parabola was known to Euclid. However, we have no mention of such points or of any of their properties until we arrive at the time of Apollonius (about the middle of the second century B.C.). He proves for us in his Conics, Book III, Props. 45–52,* the following properties of foci for central conics.

I. If Ar and $A'r'$, the tangents at the extremities of the axis of a central conic, meet the tangent at P in r and r' respectively, then

(1) rr' subtends a right angle at each focus S and S';

(2) $\angle rr'S = \angle A'r'S'$ and $\angle r'rS' = \angle ArS$.

II. If O is the intersection of rS' and $r'S$, then OP is perpendicular to the tangent at P.

III. The focal radii of P make equal angles with the tangent at that point.

IV. If from either focus as S, SY be drawn perpendicular to the tangent at any point P, the angle AYA' will be a right angle, or the locus of Y is a circle on AA' as diameter.

V. If C is the center of the conic, a line drawn through C parallel to either of the focal radii of P to meet the tangent will be equal in length to CA.

VI. In an ellipse the sum, and in a hyperbola the difference, of the focal distances from any point on the conic is equal to the axis AA'.

Apollonius does not use or mention in any way the focus for the parabola.

The next mention that we have of foci is given us by Pappus (about the end of the third century A.D.). He gives us the first recorded use and proofs of the focus-directrix definition of conics.† In addition to these Pappus has several other lemmas that have a bearing on foci. Their statement and proof will be given in section II.

But although Apollonius and Pappus have recorded for us the most notable of the properties of the foci, neither of them attached any name to the points in question. That honor was left for Johann Kepler (1571–1630). In his work Ad Vitellionem Paralipomena quibus, Astronomæ Pars Optica Traditur, Francofurti, 1604,‡ he gives a short account of the conic sections, a part of which runs as follows:

* See Conics of Apollonius, ed. Heath, pp. 113–118.

† For a discussion of these theorems see my article, Pappus, "Introductory Paper," BULLETIN, vol. 23, No. 3, p. 134.

‡ See Kepler, Opera Omnia, ed. Frisch, Frankofurti, 1859, vol. II, p. 185.

"There are among these curves certain points of especial consideration, which have a certain definition but no name, unless they usurp for name the definition or some property. For if from these points lines are drawn to the points of contact of tangents to the section, these lines make equal angles with the tangents. . . . We, because of the properties of light and the eye, from the viewpoint of mechanics shall call these points foci. We might have called them centers, because they are on the axis of the section, if authors, in the hyperbola and ellipse, were not accustomed to calling another point the center. In the circle there is one focus, the center. In the ellipse there are two foci equally distant from the center, and more removed in the more acute. In the parabola, one focus is within the section and the other may be considered either within or without the section and removed to an infinite distance from the first focus, so that if a line drawn from this '*cæcus*' focus to a point of the section will be parallel to the axis. In the hyperbola, the external focus becomes nearer the internal focus as the hyperbola becomes more obtuse."

In the above rather free translation of Kepler's remarks two things are to be noted as distinct contributions to geometry: (1) The discovery that the parabola may be considered as having two foci; (2) the formulation of the idea that parallel lines are concurrent at a point at infinity.

Following hard upon the footsteps of Kepler, Girard Desargues (1593–1662) extended the notion of the new doctrine of infinity, used the general method of projection and gave methods for determining the foci of a conic both in a plane and when the conic is in a cone. In the plane case he used the circle defined by Apollonius in his conics.* The method for determining the foci in a cone has been summarized by Chasles as follows:

Given any conic O and a cone through it. Let O' be any section of the cone. Through the vertex V pass a plane parallel to that of O' meeting the plane of O in the line ab. Take any point t on ab and let the chord of contacts of the tangents from t to O meet ab in t'. Also let rr' be any segment of ab which subtends a right angle at V. The two sets of points tt' and rr' constitute two involutions having one segment cc' in common. The polars X, X' of c and c' correspond to the axes of O'. . . . The tangents to O from the points r and the lines from r' to their several points of contact determine on

* See I (1) above.

X an involution, whose double points correspond to the foci of O', since every tangent and its normal are harmonic conjugates to the focal distances of the point of contact.*

Another interesting determination for foci was given by Maclaurin (1698–1746).† It is as follows: Let there be a conic section and let its major axis be TT', and let its auxiliary circle be drawn. Let a tangent be drawn to the conic cutting the auxiliary circle in the points A and B. Let AC be a diameter of the auxiliary circle. Let BC be drawn and let it cut TT' in F. Then F is a focus of the conic.

Poncelet (1788–1867), however, greatly extended the theory of the foci when he worked out his theory of ideal chords, and showed that all circles in a plane pass through the same two points φ and φ' on the line at infinity, and indicated the bearing of these points on the foci of conics.‡ He also shows that an analytic calculation of the foci from their definition gives not two foci for each central conic, but four, two real ones on the major or transverse axis and two imaginary ones on the minor or conjugate axis.§ Plücker (1801–68) extended the ideas of Poncelet on foci to plane curves of all orders, regarding as a focus of any curve the point of intersection of any two tangents drawn to it from φ and φ', one from each.||

A great many properties of foci and methods for determining them have been discovered in modern times. An extensive list of references bearing on this subject may be found in Taylor's Conics Ancient and Modern and in the Encyklopädie der Mathematischen Wissenschaften, Band III_2, Heft I, pages 52–56. Only one of the theorems will be noted here. It is as follows: If any cone is cut by a plane and spheres are inscribed in the cone tangent to the plane, these will touch the plane in the foci of the conic section. One would expect to find such a theorem among those enunciated by the Greeks, but it was not stated in definite form until discovered by Dandelin in 1822.¶

* For a further discussion of the work of Desargues see C. Taylor, "Conics, Ancient and Modern," Cambridge, 1881, pp. lxi and 261, where references to the original sources may be found.

† Geometrica Organica, sive descriptio linearum curvarum universalis, London, 1720, Sec. III, p. 102. See also Poncelet, Traité des Propriétés projectives des Figures, Paris, 1865, Tome I, p. 249.

‡ See Traité, sections 89–98, 453 of tome I.

§ *Annales de Mathématiques*, vol. 8 (1818), p. 222.

|| *Crelle's Journal*, vol. 10, pp. 84–91.

¶ *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-lettres de Bruxelles*, tome II, pp. 171–202.

II. *Some Theorems of Pappus which have a Bearing on Foci.*

In addition to the above mentioned theorems of Pappus relative to the subject the following ones have some interesting properties which have a bearing on the foci. They are lemmas on the books of Determinate Section of Apollonius.*

THEOREM 1: "Let there be a semicircle AEB on the diameter AB (Fig. 1), and in this the perpendiculars CE and DZ and

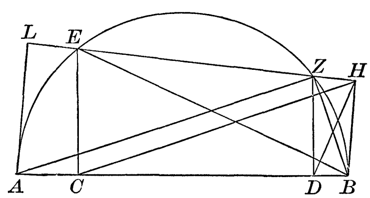

FIG. 1.

let the line EZH be drawn and to this the perpendicular BH, then three things follow: $CB \cdot BD = BH^2$, $AC \cdot DB = ZH^2$ and $AD \cdot BC = EH^2$.

"Let HC, HD, AZ and ZB be drawn. Since $\angle AZB$ is right and DZ is perpendicular to AB, then $\angle DZB = \angle BAZ$. But since the angles BDZ and ZHB are right, the points D, Z, H and B are concyclic, and $\angle DZB = \angle DHB$; then, EB being drawn, $\angle BAZ = \angle BEZ$, being angles in the same segment BZ; and in the segment BH $\angle BEZ = \angle BCH$, therefore $\angle DHB = \angle BCH$. Therefore the triangles CBH and HBD are similar and

$$CB : BH = BH : BD \quad \text{or} \quad BC \cdot BD = BH^2.$$

But also $AB \cdot BD = BZ^2$, whence if we subtract $BC \cdot BD = BH^2$, there remains $AC \cdot BD = ZH^2$. Again since $AB \cdot BC = BE^2$, if we subtract $CB \cdot BD = BH^2$, there remains $AD \cdot CB = EH^2$."

In this proof the equation $BC \cdot BD = HB^2$ signifies that HB is tangent to a circle which passes through C and D and has its center at the midpoint of EZ. If we extend EZ to L, then from the equation $AC \cdot BD = ZH^2$ we have $LE = ZH$, and the circle through D, C and H passes through L. We may then consider AB as the tangent to an ellipse whose major axis is HL, whose foci are E and Z and whose auxiliary circle is the circle through C, D, L and H, while AB is the

* See Pappus, ed. Hultsch, Book VII, Prop. 59, 61, 62 and 64.

portion of a tangent cut off between the two tangents at the vertices of the ellipse, and the circle on AB as diameter is then the circle mentioned by Apollonius as passing through the foci of an ellipse.

THEOREM 2: "Three straight lines AB, BC and CD are given (Fig. 2); then if $AB \cdot BD : AC \cdot CD = BE^2 : EC^2$,

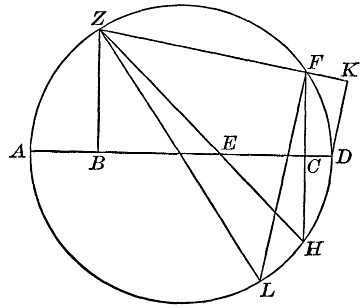

FIG. 2.

$AE \cdot ED : BE \cdot EC$ is a unique and minimum ratio and equals $AD^2 : (\sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)})^2$.

"Let there be described on the diameter AD a circle, and from the diameter let the perpendiculars BZ and CH be drawn to the circumference. Since by hypothesis $AB \cdot BD : AC \cdot CD = BE^2 : EC^2$, and since $AB \cdot BD = BZ^2$ and $AC \cdot CD = CH^2$, then $BZ^2 : CH^2 = BE^2 : EC^2$, and so $BZ : CH = BE : EC$. Therefore the triangles ZBE and HCE are similar and $\angle ZEB = \angle HEC$, and therefore the line through Z, E and H is straight. Let ZEH be drawn and let HC be produced to F, a point of the circumference, and let ZF be drawn and produced to K, whence a perpendicular KD is drawn to ZK; then by the above theorem 1, $AC \cdot BD = ZK^2$, and $AB \cdot CD = FK^2$, therefore ZF (that is $ZK - EK$) $= \sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)}$. Now let the diameter ZL be drawn, and FL also; then $\angle ZFL = \angle ECH$, and in the segment ZF, $\angle ZLF = \angle ZHF$. Therefore the triangles ZFL and ECH are similar. Therefore $LZ : ZF = HE : EC$; that is, because LZ and AD are diameters, $AD : ZF = HE : EC$, and so $AD^2 : ZF^2 = HE^2 : EC^2$; that is, on account of the similar triangles HCE and ZBE which make $HE : EC = ZE : EB$, $AD^2 : ZF^2 = HE \cdot ZE : EC \cdot EB = AE \cdot ED : BE \cdot EC$. And the ratio $AE \cdot ED : BE \cdot EC$ is unique and minimum, and we have shown above that $ZF = \sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)}$; therefore $AE \cdot ED : BE \cdot EC = AD^2 : (\sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)})^2$ is a unique and minumum ratio."

In this theorem as in the above, AD may be considered as a tangent to an ellipse which has Z and F for foci and K for a vertex. Moreover E is the point of contact of AD with the ellipse. This is evident if we notice that $\angle ZEB = \angle HEC = \angle CEF$. Pappus, however, does not prove that the ratio is actually a minimum. Fermat does this in his theory of maximum and minimum, where he remarks that Pappus recognized the difficulty of solving such problems.*

THEOREM 3: "Again let there be three given lines AB, BC and CD (Fig. 3); then if $AD \cdot DB : AC \cdot CB = DE^2 : EC^2$,

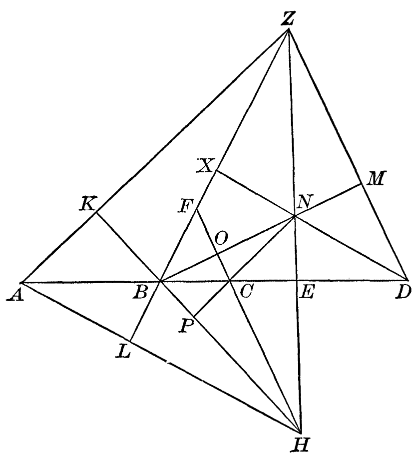

FIG. 3.

$AE \cdot EB : CE \cdot ED$ is a unique and minimum ratio and is equal to $(\sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)})^2 : DC^2$.

"Let the perpendicular EZ be drawn from E to AD and produced so that $AD \cdot DB = ZD^2$ and let HC be drawn parallel to ZD. Since by hypothesis $AD \cdot DB : AC \cdot CB = DE^2 : EC^2$ and in the similar triangles ZED and HEC, $DE : EC = DZ : CH$, then $AD \cdot DB : AC \cdot CB = DZ^2 : CH^2$; and because of the construction $AD \cdot DB = DZ^2$, $AC \cdot CB$

* See Oeuvres de Fermat, Paris, 1896, vol. 3, p. 134.

$= CH^2$. Let AZ, ZB, AH, HB be drawn and let CH be produced and cut the line BZ in F. Since $AD \cdot DB = DZ^2$, $\angle BZD = \angle ZAB$. But because $AC \cdot CB = CH^2$, or $AC : CH = CH : CB$, the angles BHC and BAH are equal. But since DZ and HF are parallel, $\angle BZD = \angle BFH$. Let HB and ZB be produced, and let HB cut AZ in K and ZB cut AH in L; then $\angle BFH + \angle BHF = \angle KBZ$. Moreover $\angle BFH = \angle BZD = \angle ZAB$ and $\angle BHF$ or $BHC = \angle BAH$; therefore $\angle ZAB + \angle BAH = \angle KBZ = \angle KAL$. But $\angle KBL + \angle KAL =$ two right angles, and so the points A, L, B, and K are on a circle. Therefore the angles at L and K are right.* Now let BM be drawn perpendicular to ZD, and let it cut the line EZ in N, and let DN be drawn and produced to X, a point of BZ. Then DX is perpendicular to ZL (see footnote below) and is perpendicular to HL. Let HF cut BN in the point O; then HO is perpendicular to BM (for ZD is perpendicular to BM, and by construction HF is parallel to ZD). Now $AC \cdot CB = CH^2$; therefore $\angle BHC = \angle HAC$. But let NC be produced to P the point of intersection with BH; then NP is perpendicular to BH. Therefore by similar triangles BOH and BPN, the points O, P, H and N are on a circle and in the segment PO $\angle PHO = \angle PNO$, or $\angle BHC = \angle CNB$. Further $\angle HAB = \angle BDN$ in the parallels HA and DX. Therefore if we note that $\angle BHC = \angle CNB$ and $BHC = \angle HAC$, and $\angle HAB = \angle BDN$, whence $\angle CNB = \angle BDN$. Therefore the triangles BNC and BDN with the common angle NBD are similar and in these $DB : BN = BN : BC$ or $DB \cdot BC = BN^2$. But since in the triangle BDZ, DNX is perpendicular, and to this the lines ZN and NB are drawn, then $ZD^2 - DB^2 = ZN^2 - NB^2$. But from construction $AD \cdot DB = ZD^2 = AB \cdot BD + BD^2$; then $ZD^2 - DB^2 = AB \cdot BD$, and so $ZN^2 - NB^2 = AB \cdot BD$. But we have proved that $NB^2 = DB \cdot BC$ and so $ZN^2 = AB \cdot BD + DB \cdot BC = AC \cdot BD$. Therefore $ZN = \sqrt{(AC \cdot BD)}$.

"Again as above $HN^2 - NB^2 = HC^2 - CB^2$. But because as above $AC \cdot CB = HC^2 = AB \cdot BC + BC^2$, $HC^2 - BC^2 = AB \cdot BC$, and so $HN^2 - NB^2 = AB \cdot BC$. But we have shown that $NB^2 = DB \cdot BC$ and so $HN^2 = AB \cdot BC + DB \cdot BC = AD \cdot BC$ or $HN = \sqrt{(AD \cdot BC)}$. Therefore $ZN + NH = \sqrt{(AD \cdot BC)} + \sqrt{(AC \cdot BD)}$.

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* Pappus proves this in a preceding lemma. The proof offers no difficulty. See Pappus, Book VII, Prop. 60.

"Again since the angle ZKH is right and AE perpendicular to ZH, by similar triangles AZE and HBE (each being similar to the triangle ABK), then $AE : EZ = HE : EB$, or $AE \cdot EB = ZE \cdot EH$. Therefore $AE \cdot EB : CE \cdot ED = ZE \cdot EH : CE \cdot ED$. But because ZD and HC are parallel, $ZE : ED = HE : EC$ and from this $ZH : CD = ZE : ED$; therefore $ZE \cdot EH : DE \cdot EC = ZE^2 : ED^2 = ZH^2 : CD^2$. Therefore $AE \cdot EB : CE \cdot ED = ZH^2 : CD^2$, and the ratio $AE \cdot EB : CE \cdot ED$ is singular and minimum, and so, since $ZH = \sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)}$, the ratio $(\sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)})^2 : CD^2$ is unique and minimum."

In the above theorem let us assume the line AD to be tangent to a parabola at the point E, and let O be the vertex of the parabola, ON the axis and Z the point in which the normal EN cuts the parabola, and let AZ be the tangent to the parabola at the point Z. Then all the relations in the proof of the above theorem will readily follow.

In theorem 64 Pappus proves that if the points A, B, C, D and E have the order $ABCDE$ on a line then $AD^2 : (\sqrt{(AC \cdot BD)} + \sqrt{(AB \cdot CD)})^2 = AE \cdot ED : BE \cdot EC$ is a unique and maximum ratio and this may be interpreted in terms of the hyperbola in exactly the same way that theorem 2 was interpreted for the ellipse.

Whether Pappus or Apollonius were thinking in terms of conic sections or not in these three problems seems rather doubtful, but if they were it is interesting to note that in the case of the ellipse and the hyperbola use is made of the foci while in the parabola no use is made of any such point. It is also interesting to note that these three are the only maximum and minimum problems (according to Pappus) in the second book of Determinate Section, and that these three cases may be made to depend upon the three conic sections.

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