32. The problem of classifying three-dimensional manifolds.
33. Particular three-dimensional manifolds. Riemann spaces. Poincaré spaces.

VIRGIL SNYDER.

NOTE ON THE ORDER OF CONTINUITY OF FUNCTIONS OF LINES.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, September 4, 1916.)

It has been proved that if a linear function of a line has continuity of the zeroth order, it can be expressed as the limit of a sequence of definite integrals,* as an integral of Stieltjes,† or as a Lebesgue integral.‡ As has been remarked by Bliss,§ many of the functions occurring in the calculus of variations do not have such continuity. The object of the present note is to show that if a function $U[y(x)]$ is linear and has continuity of the nth order, and if $y(x)$ is of class $C^{(n)}$, then $U[y(x)]$ is equal to the sum of a linear function of $d^n y(x)/dx^n$ which has continuity of the zeroth order, and a function of the values of $y(x)$ and its derivatives at an end point of the curve considered.

The proof is very simple. If $y(x)$ is of class $C^{(n)}$ it can be expressed as

$$y(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!}(x-a)^{n-i}.$$

Then, since $U[y(x)]$, is linear,

$$U\left[y\binom{b}{x}_{a}\right] = V\left[y^{(n)}\binom{b}{x}_{a}\right] + \sum_{i=1}^{n} \frac{y^{(n-i)}(a)}{(n-i)!} U\left[(x-\overset{b}{\underset{a}{a}})^{n-i}\right],$$

where the function

$$V[y^{(n)}(x)] = U\left[\int_a^x dx_1 \cdots \int_a^{x_{n-1}} y^{(n)}(x_n)dx_n\right]$$

* Hadamard, "Leçons sur le Calcul des Variations," p. 299.

† Riesz, *Annales Scientifiques de L'Ecole Normale Supérieure*, vol. 28 (1911), p. 43.

‡ Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 140.

§ Bliss, *Proc. Nat. Acad. Sciences*, vol. 1 (1915), p. 173.

is seen to be linear and continuous of the zeroth order, and $U[(x - a)^{n-i}]$ is a constant.

To illustrate this, the function $U[y(x)]$ will be taken as the first variation of the integral

$$J = \int_a^b f(x, y, y')dx,$$

when $y(x)$ is replaced by $y(x) + \epsilon\eta(x)$. That is,

$$U[\eta] = \epsilon \int_a^b (f_y\eta + f_{y'}\eta')dx.$$

If $f(x, y, y')$ and $y(x)$ are of class $C^{(2)}$, and $\eta(a) = \eta(b) = 0$, this may be written

$$U[\eta] = \epsilon \int_a^b \left(f_y - \frac{d}{dx}f_{y'}\right) \eta dx.$$

If $f(x, y, y')$ and $y(x)$ are only of class C', this equation is not valid. In this case

$$U[\eta] = V[\eta'] = \epsilon \int_a^b \left(f_{y'} - \int_a^x f_y dx\right) \eta' dx,$$

which is linear and continuous of the zeroth order. It is evident that the expression

$$f_y - \frac{d}{dx}f_{y'}$$

is the Volterra derivative of the integral J considered as a function of $y(x)$, and it is approached with the second order.* Similarly

$$f_{y'} - \int_a^x f_y dx$$

is the derivative of J considered as a function of $y'(x)$. It is approached with the zeroth order.

This can easily be extended to functions of surfaces. For convenience it will be supposed that the function $U[z(x, y)]$ is linear and continuous of the second order, and that $z(x, y)$ is of class $C^{(2)}$ and defined over the region $(0 \leqq x \leqq 1;$ $0 \leqq y \leqq 1)$. It follows immediately that

* For a definition of the order of approach see Fischer, *Amer. Jour. of Mathematics*, vol. 35, no. 4 (1913), p. 383.

$$z(x, y) = \int_0^x dx_1 \int_0^y z_{xy}(x_1, y_1)dy_1 + \int_0^y dy_1 \int_0^{y_1} z_{yy}(0, y_2)dy_2$$
$$+ \int_0^x dx_1 \int_0^{x_1} z_{xx}(x_2, 0)\, dx_2 + z_y(0, 0)y + z_x(0, 0)x + z(0, 0).$$

Proceeding as before,

$$U[z(x, y)] = V[z_{xy}(x, y)] + V_1[z_{yy}(0, y)] + V_2[z_{xx}(x, 0)]$$
$$+ z_y(0, 0)U[y] + z_x(0, 0)U[x] + z(0, 0)U[1],$$

where V, V_1 and V_2 are linear and continuous of the zeroth order and $U[y]$, $U[x]$ and $U[1]$ are constants.

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THE EQUATION OF A PLANE RATIONAL CURVE DEFINED BY PARAMETRIC EQUATIONS.

BY HERBERT WILLIAM RICHMOND.

THE explicit equation $F(x, y, z) = 0$ of a rational plane curve, defined by the parametric equations

$$x : y : z = A(t) : B(t) : C(t)$$

where $A(t)$, $B(t)$, $C(t)$ are polynomials of order n in t, is expressed by Salmon (Higher Plane Curves, § 44) as a determinant of $3n$ rows. In the April number of the BULLETIN Professor J. E. Rowe exhibits this equation as a symmetrical determinant of n rows, in which each element is a linear function of x, y, z. It has been my custom when lecturing upon algebraic geometry to obtain this form of equation as follows:

Let (x, y, z) be the coordinates of the point whose parameter is t, and let s be any value of the parameter; then

$$\begin{vmatrix} x & A(t) & A(s) \\ z & C(t) & C(s) \\ y & B(t) & B(s) \end{vmatrix} \quad (= \Delta \text{ say})$$

vanishes for every value of s. Imagine the determinant expanded and the factor $s - t$ removed. We have now a