integral in the same interval, and can be integrated term by term over this interval after being multiplied through by a function that is continuous there, then the series will converge or be summable to $f(x)$ at all points of the interval $0 < x < 1$ at which $f(x)$ is continuous; it will converge or be summable to $f(x)$ for $x = 1$ if $f(x)$ is continuous there and $l \neq 0$, or if $f(x)$ is zero for $x = 1$, and is continuous at that point and at an infinite number of points in its neighborhood; it will converge or be summable to $f(x)$ for $x = 0$ if the function is continuous at that point and at an infinite number of points in its neighborhood.*

We have

$$\varphi(x) = \sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 xf(x)F_\nu(\lambda_n, x)dx}{\int_0^1 x[F_\nu(\lambda_n, x)]^2dx}.$$

Multiplying by $xF_\nu(\lambda_n, x)$ for $n = 1, 2, 3, \cdots$ and integrating from 0 to 1, we obtain, in view of the fact that the F's form an orthogonal set,

$$\int_0^1 x[\varphi(x) - f(x)]F_\nu(\lambda_n, x)dx = 0 \quad (n = 1, 2, 3, \cdots).$$

From this equation and Lemma 4 our theorem follows at once.

UNIVERSITY OF CINCINNATI,
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SECOND NOTE ON REMOVABLE SINGULARITIES.

BY DR. W. E. MILNE.

IN a "Note on removable singularities"† the writer stated a theorem‡ concerning removable singularities for functions of several complex variables. An analogous theorem, with less restrictive hypotheses, is the following

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* The convergence or summability to $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at points where $f(x)$ has a finite jump does not follow directly from the present method. It can be obtained from the convergence or summability to $f(x)$ at points of continuity by a method given by the writer on pp. 428–429 of the paper in volume 10 of the *Transactions*, referred to above.

† BULLETIN, vol. 21 (1914), pp. 116–117.

‡ The proof there given for this theorem is incomplete, as Dr. Dunham Jackson pointed out. The gap is filled by the proof here given.

THEOREM. *Let the function* $\varphi(z_1, \cdots, z_n)$ *of* n *complex variables,* $n > 2$, *be analytic in the region* $(S) = (S_1, \cdots, S_n)$ *except for the points of a manifold* M *of such character that, when the* $n - 2$ *variables* $z_3, \cdots, z_n$ *are given any fixed values in their respective regions, the points of* M *whose last* $n - 2$ *coordinates are the above* $z_3, \cdots, z_n$ *are isolated, and thus their first two coordinates* z_1, z_2 *lead to isolated points in* (S_1, S_2). *Then the function* φ *has a limit in the points of* M, *and if defined as equal to its limit will be analytic without exception in* (S).

Proof. Let $P : (a_1, \cdots, a_n)$ be any point on M, and hold $z_3, \cdots, z_n$ fast at $(a_3, \cdots, a_n)$. Since the points in (S_1, S_2) corresponding to $(a_3, \cdots, a_n)$ (i. e., points (z_1, z_2) such that the points $(z_1, z_2, a_3, \cdots, a_n)$ are points of M) are isolated, it is possible to draw circles c_1 about a_1 and c_2 about a_2 small enough to exclude all the points thus arising except (a_1, a_2). Let t_1 be on c_1 and t_2 on c_2. Then φ will be analytic at $(t_1, t_2, a_3, \cdots, a_n)$ since this point is not on M, and hence to each (t_1, t_2) corresponds a positive number K such that $\varphi(t_1, t_2, z_3, \cdots, z_n)$ is analytic in $z_3, \cdots, z_n$ when $|z_i - a_i| < K$. Since the set of points (t_1, t_2) is closed it is easily seen that the lower limit K_0 of the corresponding set of K's is not zero, so that $\varphi(t_1, t_2, z_3, \cdots, z_n)$ is analytic in $z_3, \cdots, z_n$ when $|z_i - a_i| < K_0$, $i = 3, \cdots, n$, for *every* point (t_1, t_2) on c_1 and c_2.

Now φ may be so defined on M as to be analytic in z_1 and z_2 throughout (S_1, S_2) for every choice of $z_3, \cdots, z_n$ in $(S_3, \cdots, S_n)$, because the singularities in (S_1, S_2) are isolated, and isolated singularities for a function of two complex variables are removable. Therefore if z_1 is inside c_1 and z_2 inside c_2 we have

$$\varphi(z_1, \cdots, z_n) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_1}\int_{c_2} \frac{\varphi(t_1, t_2, z_3, \cdots, z_n)dt_1dt_2}{(t_1 - z_1)(t_2 - z_2)}.$$

The right-hand side of this equation is analytic in all n variables together at the point P, and therefore φ is analytic in all n variables at P. Since P was any point on M, the theorem is established.

Remark. Two important classes of singular manifolds to which the above theorem will apply were suggested to me by Professor Osgood.

The first is the class of manifolds defined in the neighborhood of a point by two equations

$$F_1(z_1, \cdots, z_n) = 0, \qquad F_2(z_1, \cdots, z_n) = 0,$$

where F_1 and F_2 are each analytic at the point in question and vanish there, but have no common factor there. Suppose for simplicity that the point is the origin. Then by Weierstrass's theorem of factorization it will be possible, at least after a non-singular linear transformation of the independent variables, to write

$$F_1 = [z_1^m + A_1 z_1^{m-1} + \cdots + A_m]\theta(z_1, \cdots, z_n),$$

$$F_2 = [z_1^\mu + B_1 z_1^{\mu-1} + \cdots + B_\mu]\psi(z_1, \cdots, z_n),$$

where the A's and the B's are functions of $z_2, \cdots, z_n$ which are analytic at the origin and vanish there, while θ and ψ are analytic at the origin and do not vanish there. The resultant of the polynomials in brackets is a function $R(z_2, \cdots, z_n)$ analytic at the origin and vanishing there, but not identically zero. Hence we can make a linear transformation of $z_2, \cdots, z_n$ so that $R(z_2, 0, \cdots, 0)$ is not identically zero. Now it will be seen that when $z_3, \cdots, z_n$ are all held fast at the origin the point $z_1 = 0, z_2 = 0, \cdots, z_n = 0$, is an isolated point of the manifold M. For in a sufficiently small neighborhood of the origin in the z_2-plane the function $R(z_2, 0, \cdots, 0)$ vanishes only when $z_2 = 0$; and when $z_2 = 0, z_3 = 0, \cdots, z_n = 0$, the only common solution of F_1 and F_2 in the neighborhood of the origin in the z_1-plane is $z_1 = 0$.

The second class is the class of $(2n - 4)$-dimensional real manifolds which are *regular* at the point in question (which we may again take as the origin). For then M is defined by the equations

$$z_i = \varphi_i(t_1, \cdots, t_m) + \sqrt{-1}\psi_i(t_1, \cdots, t_m),$$
$$(i = 1, 2, \cdots, n; \quad m = 2n - 4),$$

where the φ's and ψ's are single-valued real functions of the real parameters $t_1, \cdots, t_m$ which vanish at the origin and have continuous first partial derivatives there. Moreover the matrix

$$a = \left\| \begin{array}{ccc} \dfrac{\partial \varphi_1}{\partial t_1} & \cdots & \dfrac{\partial \varphi_1}{\partial t_m} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \dfrac{\partial \varphi_n}{\partial t_1} & \cdots & \dfrac{\partial \varphi_n}{\partial t_m} \\ \dfrac{\partial \psi_1}{\partial t_1} & \cdots & \dfrac{\partial \psi_1}{\partial t_m} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \dfrac{\partial \psi_n}{\partial t_1} & \cdots & \dfrac{\partial \psi_n}{\partial t_m} \end{array} \right\|$$

is of rank m at the origin. A general homogeneous linear transformation of the z's,

$$T: \qquad z_i' = \sum_{j=1}^{n} a_{ij} z_j,$$

yields a linear transformation T' of the φ's and ψ's of the form

$$T': \qquad \begin{aligned} \varphi_i' &= \sum_{j=1}^{n} (\alpha_{ij}\varphi_j - \beta_{ij}\psi_j), \\ \psi_i' &= \sum_{j=1}^{n} (\beta_{ij}\varphi_j + \alpha_{ij}\psi_j), \end{aligned}$$

where $a_{ij} = \alpha_{ij} + \sqrt{-1}\beta_{ij}$. If T' is non-singular then T is non-singular. Let δ be the matrix of the transformation T' and $\bar{a}$ the matrix corresponding to a after the transformation. Then

$$\bar{a} = \delta \times a.$$

The m-rowed determinants in $\bar{a}$ are polynomials in the α_{ij} and β_{ij} which are not identically zero, since a is of rank m. Therefore a particular transformation T' may be chosen so that all the m-rowed determinants in $\bar{a}$ are different from zero and also so that T' is non-singular when $t_1 = t_2 = \cdots = t_m = 0$. Then after the transformation T corresponding to T' has been made, if $z_3, \cdots, z_n$ are all held fast at the origin, the point $z_1 = 0$, $z_2 = 0$ is an isolated point of M.

BOWDOIN COLLEGE,
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