infinity as well as at one or more finite points. Under restrictions similar to those usually imposed, a function $f(x, y)$ may be developed in a series of solutions with Fourier coefficients.

33. Following out methods previously used by Professor W. A. Hurwitz in discussing mixed linear integral equations in one dimension, Dr. Rosenbaum in this paper establishes similar results for the case of two dimensions. The unknown function appears under integral signs operating over a plane region and over curves, and the values of the unknown function at special points also appear in the equation. The adjoint system of equations now involves as unknowns one function of two variables, several functions of one variable, and several constants. The notions of resolvent system, orthogonalization of principal solutions, and pseudo-resolvent system, receive similar generalizations.

34. The Riemann definition of the definite integral of a bounded function involves the values of the function at arbitrary points of the sub-intervals of a scheme of subdivision, while the Cauchy definition involves the values only at ends of sub-intervals. It is obvious that if the Riemann integral exists the Cauchy integral will exist; it is not immediately evident whether the converse is true or false. Professor Gillespie proves in this note that the two definitions are equivalent.

F. N. COLE,
Secretary.

INFINITE REGIONS IN GEOMETRY.

BY PROFESSOR EDWIN BIDWELL WILSON.

(Read before the American Mathematical Society, February 28, 1914.)

THE recent contribution by Professor Bôcher on "The infinite regions of various geometries"* puts me in mind of some ideas which I have long held on this subject and which I desire to offer to readers of his article.

The points which he most desires to make are:

1°, that when we are dealing with other geometries than

* This BULLETIN, vol. 20, pp. 185–200, January, 1914.

the projective we should replace the infinite line of plane projective geometry and the infinite plane of projective geometry in three dimensions by such other infinite region as may be most appropriate to the geometry we are considering, and

2°, that in particular when we introduce transformations which throw points to infinity we have not put our work in satisfactory form until we have made clear what infinite region we assume.

With the second of these points I am entirely in accord, and with the first, also, if it is properly understood; but I am inclined to think that Professor Bôcher has not put the matter of infinity in geometry in the best way* and that a literal adherence to his position would be unfortunate.

Just as he protests against the use of infinity in a carefree manner in different geometries, I wish on my part to protest against its careless use in any particular geometry. According to the view which I believe to be in fullest consonance with modern scientific attitudes toward geometry, there are no infinite (or ideal) points in the projective plane, no line at infinity in projective geometry, no infinite point in real inversive geometry, nor any pair of lines at infinity in complex inversive geometry.

To substantiate this point of view and to show at the same time the relation of my view to Professor Bôcher's I must give some definitions, in particular some definition of geometry. And as my predecessor gave no definition of what he meant by a geometry (except by implication), I shall begin by formulating a definition which seems to me to express his point of view.

Definition 1.—A geometry is the ensemble of those properties of configurations in the euclidean plane (and in its ideal extensions) which are invariant under the transformations of an r-parametered group†

* There can be no doubt that Professor Bôcher is familiar with the views which I shall expound and that he had reasons which seemed best to him for sticking close to the exposition he chose. There is still a different presentation, more elementary than either of ours, which I have heard Professor F. S. Woods offer at a semi-public gathering and which I hope he may offer to the BULLETIN; for in a subject like infinity in geometry I believe that the maximum satisfaction comes only with a multiplicity of views.

† It is only for ease that we restrict the definition to the plane and omit from the statement such groups as that of the Bewegungen und Umlegungen (motions and orthogonal reflections).

$$X' = X'(X, Y; c_1, c_2, \cdots, c_r), \quad Y' = Y'(X, Y; c_1, c_2, \cdots, c_r).$$

Definition 2.—For any particular choice of the parameters such points (X, Y) as render X' or Y' or both infinite are said to be thrown to infinity, and the infinite region of the geometry is of the type of the locus of such points (X, Y) for general values of the parameters.

Except for the unusual precision relative to infinity and a phraseology introduced by Lie, this definition is old. Applied to the characterization of projective geometry it represents the point of view early in the science of projective geometry—Poncelet, Steiner, Chasles, Salmon, Fiedler, and Cayley might reasonably be associated with it. It is still of vital pedagogic importance when we desire to lead the student for the first time from his known euclidean geometry to the new projective. For the recapitulation theory seems here to apply; the natural entogeny is an accelerated phylogeny.

Beginning, however, with von Staudt and continuing through a long succession which we may at present terminate with Veblen and Young,* there has been a tendency to place projective geometry on its own feet, to define it in terms not extraneous to itself. Although such a development has not yet reached many other geometries we are, I think, sufficiently advanced in our point of view to regard the proper definition of a geometry as something like this:†

Definition 3.—Given a system containing a set of undefined symbols (one or more classes of elements, one or more relations) and a set of primitive propositions connecting them; the geometry of this system is constituted of the body of propositions logically deduced from the primitive propositions.

* "A set of assumptions for projective geometry," *Amer. Journ. of Mathematics*, vol. 30, pp. 347–380 (1908).

† See E. V. Huntington's definition of abstract geometry on page 526 of his article "A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion," *Math. Annalen*, vol. 73, pp. 522–559 (1913). He restricts his definition to cover only the particular geometry he is expounding, but states that such a definition is applicable in other geometries also. It is entirely possible that the definition which we formulate should not be applied in general to a system as yet unformed, but should be applied anew in each particular instance after the system has been formed and then only if the person who forms the system desires to call it a geometry; for in no other way does it seem possible to include all systems which have been or may be called geometries without also including pretty much every deductive system. For restrictions see definitions by B. Russell, Principles of Mathematics, p. 372, and A. N. Whitehead, The Axioms of Projective Geometry, p. 5.

(If we desire to define a geometry as distinguished from a class of geometries, we should insist that the system be categorical.)

Definition 4.—Any class of elements selected from that class, or from one of those classes,* which enters into the definition of a geometry may be called a region of the geometry. The infinite region of the geometry would be, if it existed, a certain special region. The specialization would have to be effected by means of some property which belonged to the geometry and to which the concepts finite and infinite (in some of their many senses) were alternatively applicable.†

Why, upon the basis of Definitions 3 and 4, has the projective plane or projective geometry no infinite region? Simply because if we take any purely projective definition of projective geometry, there are no special regions whatsoever which are singled out from the rest of the plane; a fortiori there can be no infinite region. One of the things which we should be most careful to impress upon the student of projective geometry is that the projective plane is entirely homogeneous.

For a similar reason real euclidean geometry has no infinite region; it has no special points or lines in it. We may refer to Huntington's paper previously cited. The non-euclidean geometry developed in detail by Lewis and myself for the representation of the principle of relativity is also without an infinite region.‡

As to inversive geometry it may well be that as yet there is no exact formulation in postulates, but the possibility of such formulation is so evident that it is safe to say that inversive geometry has no infinite element or infinite region; the inversive plane is homogeneous. We might as well maintain that the surface of a sphere or spherical geometry, when defined by a system of postulates appropriate to the geometry,§

* We could define mixed regions by collecting elements from different classes.

† We formulate no precise definition of infinite elements or infinite region because in most of the geometries which have been handled in the modern logical manner there is no infinite region.

‡ Wilson and Lewis, "The space-time manifold of relativity," *Proceedings of the American Academy*, vol. 48, pp. 389–507. There are singular loci in the geometry, but no special points. If we consider as fundamental the right line and angle, we could regard the singular lines as infinite elements; but the advantage of such procedure is problematical.

§ For a definition which is not proved to be either complete or categorical see E. B. Wilson, "Seven lectures on spherical geometry," *Amer. Math. Monthly*, serially in 1904. In those lectures I made use of the idea of a possessed a special element (such as a north pole) as that the inversive plane had a special element such as the infinite point.*

As now, speaking from the geometric and logical viewpoints, we have abolished the infinite regions in some, and could abolish them in all, of the geometries of which Professor Bôcher speaks, it is necessary to answer the question: What and where are the infinite regions of which he speaks? They are ideal, they are perhaps nowhere. They arise algebraically through the becoming infinite of some function; this is accidental to our choice of coordinates.† They originate geometrically from the breaking down of a correspondence between the planes of two different geometries;‡ this latter is the really essential geometric fact.

group of transformations in formulating the axioms, just as Lewis and I later adopted that point of view in our geometry of relativity, loc. cit. In view of the importance of the group concept in modern geometry it might be desirable that some of our eminent specialists in postulates should construct systems in which that concept was emphasized. It is unfortunate in some respects that groups have been tied so closely to analytical representations as far as their geometric uses go.

* Professor Bôcher in a footnote calls attention to the fact that in Study's long treatment of Das Apollonische Problem, *Math. Annalen,* vol. 49, pp. 497–542, there is not a word said concerning the nature of the infinite region. According to our reading of Study's work the reason that no mention of the infinite region occurs may well be that for him, as for us, there is no such region in the geometry of inversion. We may be reading our own ideas into Study's text (which would be a heinous offence on our part toward so illustrious a geometer), but we believe that he makes his point of view quite clear. He does not set up a categorical system of postulates for inversive geometry, but he does point out very precisely that one of the chief differences between his geometry and Mascheroni's is that the latter uses the center of a circle whereas he makes no use of it. As one of Study's fundamental constructions is to find the inverse of any point with respect to any circle, the center could be found if the point at infinity were in the system, and then there would be but little gain in banishing the use of the center. Furthermore Study makes very strong the point of view which actuates him in this article, as in other of his work, namely, the desire to remain completely within his geometry.

† It is interesting to observe that we cannot represent the points on a simple closed curve by a continuously varying parameter so that the correspondence shall be one-to-one. What we do is to use a parameter t subject to $t_0 \leqq t \leqq t_1$, which assigns to the same point the values t_0 and t_1 (to drop t_0 or t_1 from the interval would be to render the correspondence discontinuous). Or we write $-\infty < t < \infty$ and identify ∞ and $-\infty$, and thus introduce the singularity ∞. This is no singularity of the curve. It is merely an unhappy lack of correspondence between arithmetic and geometric types of order. H. B. Phillips and C. L. E. Moore in their "Algebra of plane projective geometry," *Proceedings of the American Academy,* vol. 47, pp. 737–790, probably would have been glad to avoid infinity; but with a non-homogeneous algebra that was out of the question.

‡ One of these being euclidean in the cases Professor Bôcher discusses.

If we start with non-homogeneous (rectangular cartesian) coordinates and write down the equations of a general linear transformation, we get introduced to infinity through the vanishing of the denominator which is common to the two fractions. However, this means merely that from the point of view of projective geometry and the projective plane, we have made a very injudicious start—howsoever judicious the start may have been for a modulation of a pedagogical nature from metric to projective geometry for the student first approaching the latter. Had we selected a triangle with a unit point, introduced trilinear coördinates based on a projective number system, and used the homogeneous form of the transformation, such as is almost invariably used in invariant theory, we should never have met any infinity.

We believe in the pedagogic modulation, but we believe also that particular stress should be laid on the fact that the ideal elements which are introduced are ideal elements of the extended euclidean plane and of extended euclidean geometry rather than of projective geometry, and that they are introduced in or, better, adjoined to the euclidean plane for the purpose of bringing about a correspondence (one out of infinitely many) between the euclidean and projective planes, not only for pedagogic purposes, but rather especially for the sake of carrying across theorems from either geometry into the other. The process of throwing some line, a perfectly normal line of the projective plane (we cannot say a finite line because there is no distinction of finite and infinite), into the ideal region of the euclidean plane is of great use in saving a new demonstration of certain theorems.

When we turn to circle geometry a similar state of affairs is found. We are able to set up a correspondence between the euclidean and circular planes which is one-to-one, points corresponding to points and circles to circles (with proper qualifications), except that there is an extra point in the circular plane for real geometry and two extra imaginary lines intersecting in a real point in the case of the complex circular geometry. We promptly adjoin these as ideal elements in the euclidean plane for the sake of perfecting the correspondence—the reasons being as before partly pedagogic, partly lexicographic.*

* It would indeed be an interesting study in euclidean geometry to take the general solution of the Apollonian problem as developed in inversive

It is somewhat doubtful whether we should regard the ideal regions thus adjoined to the euclidean plane as lying in that plane; it may be better to regard them as lying nowhere at all. Two reasons for this occur to anybody at once. First, the postulates upon which euclidean geometry has been built up are in many cases no longer true for the extended euclidean plane. The introduction of the ideal elements into the euclidean plane has simplified certain statements, namely, such as are essentially projective or inversive (as the case may be), but it has greatly complicated others, which are essentially euclidean. Second, the two cases we have considered show that different ideal regions have to be adjoined in different cases, and that these different ideal regions are mutually incompatible so that they cannot coexist.

The problem of correspondence between the projective and the euclidean planes or between the inversive and euclidean planes is simple by virtue of the fact that the planes do not really clash, in each case we have merely to remedy a defect in the euclidean plane. A vital reason, too, for the naturalness of the correspondence lies in the fact that the euclidean group

$$p, \quad q, \quad yp - xq$$

is a subgroup of the projective group

$$p, \quad q, \quad yp - xq, \quad yp + xq, \quad xp + yq, \quad xp - yq,$$
$$x^2p + xyq, \quad xyp + y^2q,$$

and of the inversive group

$$p, \quad q, \quad yp - xq, \quad xp + yq,$$
$$2xyp + (y^2 - x^2)q, \quad (x^2 - y^2)p + 2xyq.$$

When we try to establish a correspondence between two geometries which are not related in such a manner that one is a subgroup of the other, the matter is not so simple. How could we set about mapping the projective plane on the inversive plane or vice versa? It is futile to content ourselves with trying to map the points of one upon the points of the other; we must be able to carry over certain configurations or

geometry and see how many special cases we might get by selecting different points of the figure for relegation to infinity, and the Apollonian problem for euclidean geometry is not solved until all possibilities have been enumerated.

relations. But projective geometry deals with lines, conics, and so on, whereas inversive geometry deals with circles, cyclides, and so forth. As a problem in abstract geometry, in logic, it appears somewhat difficult to set up a satisfactory and useful relationship between the planes.*

We may overcome the difficulty very readily by coming down to the common subgroup, to euclidean geometry,† and make the transfer between projective and inversive geometry by the intermediary of euclidean. We notice now, however, that the projective and inversive planes clash in their requirements for extensions of the euclidean plane and we shall be on more comfortable ground if we keep the intermediary instead of attempting to cast it aside and obtain a direct correspondence. Projective geometry can be forced into the inversive mould, or inversive geometry into the projective mould only by cracking the mould or the geometry; but if we insist on making the correspondence direct, it would be difficult to say why we should locate the crack at infinity in either geometry.

The definitions of geometry and of infinite region which we have attributed to Professor Bôcher (Definitions 1 and 2) suggest at once the methods of Sophus Lie, and it is an interesting question to ask whether in plane geometry we are bound to introduce for different groups other regions at infinity than those which arise in the projective and inversive groups. Now Lie has tabulated the varieties of groups which occur in plane geometry in twenty-six entries.‡ We may integrate and determine the finite equations of the group. I have done this in a great many cases and have not found any other types of region.§ In space|| we have a considerable variety—a plane,

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* Abstractly a similar difficulty exists in the case of the projective (or inversive) plane and the euclidean plane; for the projective and euclidean lines are different, the inversive and euclidean circles are not the same—but the differences are not so serious, there are marked similarities as well as differences.

† Indeed to what Klein calls the Hauptgruppe, p, q, $yp - xq$, $xp + yq$, which contains similitude transformations; but euclidean geometry is more familiar.

‡ Continuierliche Gruppen, p. 360.

§ This does not mean that in all similar groups, that is, in all reducible to a common type by a proper choice of variables, the infinite region would be the same; it merely means that a choice of variables exists for which the infinite region becomes as indicated.

|| Transformationsgruppen, vol. 3, pp. 122–178.

two planes intersecting in a line, three planes with a point in common, a quadric cone,* and perhaps other cases.

We have only one or two more illustrations to give toward substantiating our point of view that the introduction of infinity as Professor Bôcher does is not entirely satisfactory from the geometric point of view, and indeed violates established nomenclature.

Suppose that we consider the group

$$yp - xq, \quad x^2p + xyq - p, \quad xyp + y^2q - q.$$

This is a projective subgroup which leaves the circle $x^2 + y^2 - 1 = 0$ invariant, and is closely associated with Lobachevskian geometry.† According to ordinary nomenclature the fixed circle is the absolute and the absolute is at infinity. Not so, however, if we must determine the finite equations of the group and see what nature of locus is relegated to the bourne beyond the euclidean plane.‡

Suppose that we consider the group

$$yp - xq, \quad x^2p + xyq + p, \quad xyp + y^2q + q.$$

This is a projective subgroup§ which leaves $x^2 + y^2 + 1 = 0$ invariant, and is closely associated with Riemannian geometry. According to ordinary nomenclature lines in this geometry are closed and have a finite length (at least in the real plane) and there does not arise the question of infinity. Not so, however, if again we must determine the finite equations of the group and observe what manner of locus it is that yields infinite values for the transformed coördinates.

The case of real Lobachevskian geometry is illuminating. When that geometry is defined by a set of postulates (see Coolidge, Non-Euclidean Geometry) there is no infinite

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* If we may call the minimum cone in inversive space geometry a quadric cone. Can we appropriately, except in euclidean geometry, call it a minimum cone?

† See Klein-Fricke, Automorphe Funktionen. We can also, as Klein has pointed out, use a subgroup of inversive geometry; this is conformal but lines become circles orthogonal to a fundamental circle.

‡ This line lies in what is called the transfinite or ultra-infinite (as contrasted with finite or infinite) region. See Coolidge, Non-Euclidean Geometry, p. 85.

§ With an appropriate change of variable, the group may be made a subgroup of the inversive group, as suggested in the second footnote above. Indeed the groups which leave $x^2 + y^2 \pm 1 = 0$ invariant are

$$yp - xq, \quad (x^2 - y^2)p + 2xyq + p, \quad 2xyp + (y^2 - x^2)q \pm q.$$

region at all, and to modify the postulates so as to introduce any points at infinity would be an unpleasant complication resulting in no gain to the geometry. When, however, we desire to map the projective (or inversive) and Lobachevskian planes one upon the other, we find that the Lobachevskian plane lies entirely within a conic of the projective plane (and entirely upon one side of a circle in the inversive plane—there is here no distinction between inside and outside). To perfect the correspondence we adjoin to the Lobachevskian plane the conic (or circle) as an infinite region and the region outside the conic (or upon the other side of the circle) as an ultra-infinite region.

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FAMOUS PROBLEMS OF GEOMETRY.

"*Squaring the Circle,*" *A History of the Problem.* By E. W. HOBSON. Cambridge, at the University Press, 1913. iii+57 pp. Price 3s.

A FASCINATING and voluminous volume could be written on ancient problems of geometry, their influence on the progress of mathematics and the various developments in mathematics which contributed to their generalization or final settlement.

There is the familiar problem,* to draw from a given point P a line such that the line segment cut off by two intersecting lines l_1, l_2 shall be of given length. This problem is capable of solution with ruler and compasses in but one case, namely when P is on a bisector of an angle between l_1 and l_2. Suppose this condition to obtain. The problem is not an easy one, in general, but Apollonius (about 225 B.C.), known to his contemporaries as the "great geometer," found an elegant solution.† The complete discussion for the case of l_1 and l_2 at right

* Cf. my paper "Discussion and history of certain geometrical problems of Heraclitus and Apollonius," *Proc. Edinb. Math. Soc.*, vol. 28 (1909–1910), pp. 152–178.

† Although twice proposed in the *American Math. Monthly* (Feb., 1910, vol. 17, p. 48, cf. pp. 140–141; Feb., 1911, vol. 18, p. 44, cf. pp. 114–115) no solution has been forthcoming.