All the steps we have here taken are justified since $v(x, \lambda)$ and $v'(x, \lambda)$ are both continuous in (x, λ) and analytic in λ.*

We have thus established Picone's result:

The differential system

$$\frac{d}{dx}(ku') + \lambda g u = 0, \qquad (k > 0)$$

$$u'(a) = 0, \qquad u'(b) = 0$$

in which g changes sign in ab has, if $\int_a^b g dx = 0$, no characteristic number other than zero for which the characteristic function does not vanish, otherwise it has just one such characteristic number, namely a positive one if $\int_a^b g dx < 0$, a negative one if $\int_a^b g dx > 0$.

I note in closing that the case $\int_a^b g dx = 0$ is of interest as giving one of the simplest examples of a characteristic number ($\lambda = 0$) whose order of multiplicity when regarded as a root of the characteristic equation (2 in this case) is not equal to its index (1 in this case), i. e., the number of linearly independent characteristic functions corresponding to it.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
July 15, 1914.

ON APPROXIMATION BY TRIGONOMETRIC SUMS.

BY PROFESSOR T. H. GRONWALL.

(Read before the American Mathematical Society, December 31, 1913.)

IN his paper "On approximation by trigonometric sums and polynomials"† Dr. Jackson has shown that, $f(x)$ being a function of period 2π and satisfying the Lipschitz condition

$$| f(x_2) - f(x_1) | \leqq \lambda \, | x_2 - x_1 |$$

* For the fundamental properties of this important class of functions see my paper "On semi-analytic functions of two variables," *Annals of Mathematics*, 2d ser., vol. 12 (1910), p. 18. I was not aware when I published this article that some of these properties had been already given by Dini, *Annali di Matematica*, ser. 3, vol. 12 (1906), p. 179.

† *Transactions*, vol. 13 (1912), pp. 491–515.

for all values of x_2 and x_1, then there exists, for every integer n, a trigonometric sum of order not exceeding n

$$T_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx,$$

approximating $f(x)$ in such a way that for all values of x

$$| f(x) - T_n(x) | < \frac{4J'_m}{J_m} \cdot \frac{\lambda}{n},$$

where the integer m is determined by the condition $2m - 2 \leqq n < 2m$, and

$$J_m = \frac{1}{m^3} \int_0^{\pi/2} \left(\frac{\sin mu}{\sin u} \right)^4 du, \qquad J'_m = \frac{1}{m^2} \int_0^{\pi/2} u \left(\frac{\sin mu}{\sin u} \right)^4 du. \tag{1}$$

By asymptotic considerations, Dr. Jackson shows that for $m \geqq 4$, $n \geqq 6$,

$$\frac{4J'_m}{J_m} \leqq 2.90 \cdots \tag{2}$$

It is the purpose of the present note to show that the quotient in (2) decreases as m increases,

$$\frac{J'_{m+1}}{J_{m+1}} - \frac{J'_m}{J_m} < 0 \qquad (m = 1, 2, 3, \cdots). \tag{3}$$

Since it may be shown that*

$$J_m = \frac{\pi}{3} \left(1 + \frac{1}{2m^2} \right), \tag{4}$$

the inequality (3) is equivalent to

$$\Delta_m = (2m^2 + 1) \int_0^{\pi/2} u \frac{\sin^4 (m + 1)u - \sin^4 mu}{\sin^4 u} du - (4m + 3) \int_0^{\pi/2} u \left(\frac{\sin mu}{\sin u} \right)^4 du < 0. \tag{5}$$

* T. H. Gronwall, "On the degree of convergence of Laplace's series," *Transactions*, vol. 15 (1914), pp. 1–30.

We shall begin by replacing u by an approximate expression in terms of $\sin u$. We have

$$\frac{d}{du}\,\frac{u-\sin u}{\sin^3 u}=\frac{\sin u-3u\cos u+2\sin u\cos u}{\sin^4 u}$$

and

$$\frac{d}{du}(\sin u-3u\cos u+2\sin u\cos u)=3\sin u(u-\sin u)$$
$$+(1-\cos u)^2>0 \text{ for } 0<u<\pi/2;$$

therefore $\sin u-3u\cos u+2\sin u\cos u>0$ for $0<u<\pi/2$, and consequently $\frac{u-\sin u}{\sin^3 u}$ increases monotonely with u for $0<u<\pi/2$, so that

$$u=\sin u+\eta(u)\sin^3 u,$$

(6) $$\frac{1}{6}=\eta(0)<\eta(u)<\eta\left(\frac{\pi}{2}\right)=\frac{\pi}{2}-1 \text{ for } 0<u<\frac{\pi}{2}.$$

The expression (5) for Δ_m now gives

(7) $$\begin{aligned}\Delta_m={}&(2m^2+1)\int_0^{\pi/2}\frac{\sin^4(m+1)u-\sin^4 mu}{\sin^3 u}\,du\\&-(4m+3)\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du\\&+(2m^2+1)\int_0^{\pi/2}\eta(u)\,\frac{\sin^4(m+1)u-\sin^4 mu}{\sin u}\,du\\&-(4m+3)\int_0^{\pi/2}\eta(u)\,\frac{\sin^4 mu}{\sin u}\,du\\<{}&(2m^2+1)\int_0^{\pi/2}\frac{\sin^4(m+1)u-\sin^4 mu}{\sin^3 u}\,du\\&-(4m+3)\int_0^{\pi/2}\frac{\sin^4 mu}{\sin^3 u}\,du\\&+(2m^2+1)\left(\frac{\pi}{2}-1\right)\int_0^{\pi/2}\frac{\sin^4(m+1)u-\sin^4 mu}{\sin u}\,du\\&-\frac{4m+3}{6}\int_0^{\pi/2}\frac{\sin^4 mu}{\sin u}\,du.\end{aligned}$$

Proceeding to evaluate our various integrals, we find, integrating by parts,

$$\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du = \int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,d(-\cot u)$$

$$= \int_0^{\pi/2} \cot u \frac{d}{du}\left(\frac{\sin^4 mu}{\sin u}\right) du$$

$$= \int_0^{\pi/2} \frac{4m \sin^3 mu \cos mu \cos u}{\sin^2 u}\,du$$

$$- \int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du + \int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du,$$

or

$$\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u}\,du = 2m \int_0^{\pi/2} \sin^3 mu \cos mu \, d\left(-\frac{1}{\sin u}\right)$$

$$+ \frac{1}{2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du$$

$$= 2m \int_0^{\pi/2} \frac{1}{\sin u}\frac{d}{du}\,(\sin^3 mu \cos mu)du$$

$$+ \frac{1}{2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u}\,du$$

$$= 2m^2 \int_0^{\pi/2} \frac{\sin^2 2mu - \sin^2 mu}{\sin u}\,du$$

$$+ \frac{1}{2}\int_0^{\pi/2} \frac{\sin^2 mu - \frac{1}{4}\sin^2 2mu}{\sin u}\,du.$$

Now the identity

$$\frac{\sin^2 nu}{\sin u} = \sum_{\lambda=0}^{n-1} \sin\,(2\lambda + 1)u$$

gives

$$\int_0^{\pi/2} \frac{\sin^2 nu}{\sin u}\,du = \sum_{\lambda=0}^{n-1} \frac{1}{2\lambda + 1},$$

and consequently

$$
(8) \qquad \begin{aligned}
\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u} du &= 2m^2 \sum_{\lambda=m}^{2m-1} \frac{1}{2\lambda+1} + \frac{1}{2}\sum_{\lambda=0}^{m-1} \frac{1}{2\lambda+1} \\
&\qquad - \frac{1}{8}\sum_{\lambda=0}^{2m-1} \frac{1}{2\lambda+1}, \\
\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u} du &= \sum_{\lambda=0}^{m-1} \frac{1}{2\lambda+1} - \frac{1}{4}\sum_{\lambda=0}^{2m-1} \frac{1}{2\lambda+1}.
\end{aligned}
$$

From these equations and the last part of (7) we obtain

$$
\begin{aligned}
\Delta_m < &- 2(m^2-2m-1)\sum_{\lambda=m}^{2m-1}\frac{1}{2\lambda+1} \\
&- \frac{2}{3}(4m+3)\left(\sum_{\lambda=0}^{m-1}\frac{1}{2\lambda+1} - \frac{1}{4}\sum_{\lambda=0}^{2m-1}\frac{1}{2\lambda+1}\right) \\
&+ (2m^2+1)\left(\frac{\frac{\pi}{2}-1}{2m+1} + \frac{1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1} - \frac{1}{4}\cdot\frac{1}{4m+3}\right).
\end{aligned}
$$

For $m > 2$, the first term in the expression on the right is negative, and since the difference of the two sums in the second term obviously increases with m, we have for $m > 2$

$$
\sum_{\lambda=0}^{m-1}\frac{1}{2\lambda+1} - \frac{1}{4}\sum_{\lambda=0}^{2m-1}\frac{1}{2\lambda+1} > 1 + \tfrac{1}{3} - \tfrac{1}{4}(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}) = \tfrac{32}{35};
$$

furthermore

$$
\begin{aligned}
\frac{\pi-2}{4m+2} + \frac{1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1} - \frac{1}{4}\cdot\frac{1}{4m+3} \\
< \frac{\pi-2+1-\frac{1}{4}\left(\frac{\pi}{2}-1\right)}{4m+1} < \frac{2}{4m+1},
\end{aligned}
$$

so that finally, for $m > 2$,

$$
\Delta_m < -\tfrac{2}{3}\cdot\tfrac{32}{35}(4m+3) + \frac{4m^2+2}{4m+1} < 0,
$$

which proves our theorem for $m > 2$. For $m = 1$ and $m = 2$ it is readily verified from the numerical values of J_m and J'_m given by Jackson.

From (1), (6) and (8) it follows that

$$J'_m = \frac{1}{m^2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin^3 u} du + \eta' \cdot \frac{1}{m^2}\int_0^{\pi/2} \frac{\sin^4 mu}{\sin u} du,$$

$$\tfrac{1}{6} < \eta' < \left(\frac{\pi}{2} - 1\right),$$

and consequently

$$J'_m = 2 \sum_{\lambda=m}^{2m-1} \frac{1}{2\lambda + 1} + O\left(\frac{\log m}{m^2}\right),$$

$$\lim_{m=\infty} J'_m = \lim_{m=\infty} \sum_{\lambda=1}^{2m-1} \frac{2}{2\lambda+1} = \int_0^1 \frac{dx}{1+x} = \log 2.$$

Using this result in connection with (4), it is seen that

$$\lim_{m=\infty} \frac{4J'_m}{J_m} = \frac{12 \log 2}{\pi} = 2.648 - ,$$

so that, using the numerical values of J, and J',, we may finally state the result

$$2.758 - = \frac{4J_4'}{J_4} > \frac{4J_5'}{J_5} > \frac{4J_6'}{J_6} > \cdots > 2.648 - .$$

PRINCETON UNIVERSITY,
December 20, 1913.

NOTE ON THE ROOTS OF ALGEBRAIC EQUATIONS.

BY PROFESSOR R. D. CARMICHAEL AND DR. T. E. MASON.

(Read before the American Mathematical Society at Chicago, April 10, 1914.)

1. LANDAU* has established certain interesting inequalities concerning the least root of a class of algebraic equations, having been led to these results by considerations connected with his remarkable extension and generalization of Picard's famous theorem to the effect that an entire function which fails to assume two values is a constant. These special in-

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* *Annales de l'École Normale Supérieure* (3), vol. 24 (1907), pp. 179–201; *Vierteljahrsschrift der Naturf. Gesellschaft in Zürich,* vol. 51 (1906), pp. 252–318.