

author has derived any ideas or assistance. The fourth volume completes the list of papers published up to the end of 1910: we hope that as further material accumulates it may be cast into the same form.

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### MATHEMATICAL ECONOMICS.

*Manuel d'Economie politique.* Par VILFREDO PARETO. Paris, Giard et Brière, 1909. 695 pp.

IN the year 1906 Pareto's *Manuale di Economia politica con una Introduzione alla Scienza sociale* was published at Milan. Three years later a French translation by Alfred Bonnet, with revisions by the author, appeared as volume 38 in the *Bibliothèque internationale d'Economie politique*. This work, although written by the economist who has most insistently stood for the scientific mathematical method in economics, is not itself primarily mathematical except in spirit. The text to be sure, uses curves freely, is replete with logical keenness, and draws considerably upon mathematical language and upon mechanical, or even thermodynamical, analogy in discussing and illustrating economic equilibrium; but the strictly mathematical treatment, which might be technical to the point of causing some of the world's best literary economists insuperable difficulties, has in every instance been relegated to the long appendix of 133 pages. This arrangement has also been adopted by Irving Fisher in his recent work on *The Purchasing Power of Money*. So long as only a few students of political economics acquire the necessary knowledge of calculus such a segregation must remain inevitable.

In the first place, as there seems to be no very widespread notion among mathematicians, perhaps even among economists, as to what mathematical economics is and does, it may be well to define a little.

It is clear that the individual as a social unit and the state as a social aggregate require a certain modicum of mathematics, some arithmetic and algebra, to conduct their affairs. Under this head would fall the theory of interest, simple and compound, matters of discount and amortization, and, if

lotteries hold a prominent place in raising moneys, as in some states, questions of probability must be added. As the state becomes more highly organized and more interested in the scientific analysis of its life, there appears an urgent necessity for various statistical information, and this can be properly obtained, reduced, correlated, and interpreted only when the guiding spirits in the work have the necessary mathematical training in the theory of statistics. (Figures may not lie, but statistics compiled unscientifically and analyzed incompetently are almost sure to be misleading,\* and when this condition is unnecessarily chronic the so-called statisticians may well be called liars.) The dependence of insurance of various kinds on statistical information and the very great place which insurance occupies in the modern state, albeit often controlled by private corporations instead of by the government, makes these theories of paramount importance to our social life.

As all these subjects of study are generally treated at our universities by the departments of political economy, if indeed they are treated at all, it might be natural to classify them as mathematical economics. And one doubtless recalls that M. Cantor adopts (from Petty?) the title *Politische Arithmetik* for his popular little text on such matters. But the term mathematical economics as Pareto† uses it, and as it is generally used, implies something very different, something much more akin to theoretical economics, just as the term mathematical physics does not denote the use of mathematics to reduce physical observations or even to give the theory of special physical apparatus so much as it connotes general theoretical physics. It is evident to every physicist who lifts his nose from the grindstone of his lengthy experiments, who frees and rests his mind from the oftentimes exasperatingly human perversities of his instruments, that it is only with the aid of mathematics that the salient facts of physics can be adequately set forth and correctly correlated, the underlying concepts and suppositions critically analyzed and properly refined. It is evident that notwithstanding the magnificent genius of Faraday, electromagnetism needed

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\* Those who have followed critically the government crops reports, emanating from the Department of Agriculture, have not failed at times to complain that better and plainer results might somehow be obtained.

† We would not imply that Pareto has not been interested in statistical questions; his researches in these lines are well known.

its Maxwell and its Lorentz, that howsoever wondrously Regnault might measure, thermodynamics needed its Gibbs and its Planck.

All this is true of economics, but here it is not true that every economist sees it. So intricate are the interrelations of economic phenomena, so hopeless are any extensive experimental methods, so fragmentary are as yet the desirable statistics, and so far removed from the possibility of accurate quantitative formulation does the subject seem to be, that perhaps the majority of economists, doing their best to grope their way through a fog of details and a mist of prejudices, feel it to be impossible, or at best a useless work of supererogation, to introduce mathematical methods into economics. Now, to be sure, if economists expect of the application of the mathematical method any extensive concrete numerical results, and it is to be feared that like other non-mathematicians all too many of them think of mathematics as merely an arithmetical science, they are bound to be disappointed and to find a paucity of results in the works of the few of their colleagues who use that method. But they should rather learn, as the mathematicians among them know full well, that mathematics is much broader, that it has an abstract quantitative (or even qualitative) side, that it deals with relations as well as numbers, and that it has been defined as the science which draws necessary conclusions.\*

There is no doubt that economists draw conclusions. Of these some are tenable, some are not. A few of those tenable are necessary, many are not. Indeed it is inevitable, and for a long time will remain so, that economics, especially in its applications, should be an art, and that its conclusions, far from being necessary, should largely be a matter of the opinion of the individual economist. But just because the way is dark with detail and shrouded in prejudice and impossible of complete clarification, is that any reason why any one should scorn whatever illumination mathematics can bring, whatever precision of statement it may suggest, whatever necessary conclusions it offers? If one knows precisely what his premises are and just how much he can draw from them, is he not the better off for exercising his judgment when strict deduction is

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\* For a discussion of what mathematics is, see M. Bôcher, "The fundamental conceptions and methods of mathematics," in this *BULLETIN*, vol. 11, pp. 115-135.

no longer possible? So it seems to us, and we must admit that when we read anything that Pareto writes, whether a short article in a popular review or an essay\* in the mathematical encyclopedia or a vast manual of political economy, we are agreeably impressed not alone with the cogency of the proofs, but even more with the soundness of the judgments.

Two quotations may be of interest as showing how vitally economists differ as to the use of mathematics in their field.

“Mathematics has influenced the form and the terminology of the science, and has sometimes been useful in analysis; but mathematical methods of reasoning, in their application to economics, while possessing a certain fascination, are of very doubtful utility.”†

“The effort of the economist is to *see*, to picture the interplay of economic elements. The more clearly cut these elements appear in his vision, the better; the more elements he can grasp and hold in his mind at once, the better. The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines. The old phantasmagoria disappear. We see better. We also see further.”‡

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\* We use this word advisedly; for it does not seem that Pareto's 50-page article (with apparently more to follow) in the new, French, edition of the mathematical encyclopedia is other than an essay, excellent in itself, but as an encyclopedia article on mathematical economics decidedly inferior to his earlier 25-page presentation of the same subject in the German edition. However, the whole encyclopedia, whether German or French edition, seems of late to have run riotously and fruitlessly to leaves.

† From the *Encyclopedia Britannica*, 11th edition, vol. 8, p. 902, in the long article on Economics which is signed W. A. S. H. As this signature is not reproduced in the key at the beginning of the volume, it is difficult to identify the author, but as this one sentence is his only reference to the vast literature of mathematical economics we are forced to believe that the editors of the new edition of the *Britannica* have in this instance, as in some others we have observed, let out an important subject to a man who, however eminent he may be, has not that complete command of his subject and that full sympathy with all its phases which are necessary and which alone bring competence for writing with the poise, dignity, and authority appropriate to a great encyclopedia. We should have had much more confidence in an article signed F. Y. E. On the other hand we must admit that W. A. S. H. is, in comparison with some others, extremely inoffensive and even gracious toward his mathematical confrères.

‡ From p. 119 of Irving Fisher's "Mathematical investigations in the theory of values and prices," *Transactions of the Connecticut Academy*, vol. 9 (1892), pp. 1-124. We cannot cite this work without commending it to all for the clearness and precision with which the economic problem is stated, for the keen appreciation of the limitations, as well as the advantages, of the use of mathematics in economics or other sciences, and

The foregoing paragraphs may seem rather remote from a review of the book under discussion; but they are not. For one of the things which the author makes clearest, by constant reference to it at all stages of his work, is the scientific attitude which he is taking, the attitude which he believes should be taken toward theoretical economics. He nowhere claims to do the impossible, he does not desire to overrate the usability of mathematics in applied economics, but he does insist upon an analysis, as thorough as possible, of the underlying foundations of economic theory. He is even willing to minimize the value of mathematics in the discussion of the simpler cases, where the reasoning may readily be carried out in words,\* for the sake of emphasizing the fact that it is the interdependence of the totality of economic phenomena which brings in the complications necessitating the introduction of mathematics.

Let us begin now with the appendix. We consider an idealized individual, the homo œconomicus, who lives in the presence of and is possessed of certain goods and whose actions will result from his desire to obtain the greatest pleasure from the appropriation or "consumption" of the goods. If he is in possession of given amounts  $x, y, \dots$  of different goods, he will generally be equally satisfied, neither more nor less, by certain other combinations of amounts. If his possessions be represented by a point in a space of as many dimensions as the number of goods considered, the totality of the points which represent equal satisfaction will constitute a surface of one dimension less in that space. This surface is a surface of indifference, that is, of equal desire or pleasure. By considering that he could have a choice between some different combination and its totality of equivalents, another surface could be drawn. Thus the space is ruled off with a family of

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for the critical discussion of the case against mathematics as presented by its opponents. The readers of Pareto's manual and of Fisher's monograph cannot fail to remark the similarity between the scientific points of view of the two authors.

\* For instance on p. 557 he says: Nous estimons que l'usage des mathématiques, pour des problèmes du genre de celui d'un individu et de deux, ou même de plusieurs biens, ne donne pas des résultats dont l'importance puisse se comparer à ceux que l'on obtient dans les cas de l'équilibre économique général. A notre avis c'est l'interdépendance des phénomènes économiques qui nous oblige à faire usage de la logique mathématique. Cette manière de voir peut être bonne ou mauvaise; mais en tout cas elle ne doit pas être confondue avec celle des économistes qui établissent des théories en négligeant précisément cette interdépendance.

surfaces of indifference

$$f(x, y, \dots; I) = 0 \quad \text{or} \quad I = \psi(x, y, \dots) \quad \text{or} \quad I = F[\psi(x, y, \dots)],$$

where the parameter  $I$  may be taken as the index of pleasure or ophelimity.\*

We may now pass by differentiation to the equation

$$dI = F'\psi_x dx + F'\psi_y dy + \dots = 0,$$

which is the differential equation of the surfaces of indifference. This equation merits a comment. The negative partial derivatives such as

$$-\left(\frac{dy}{dx}\right)_{y, \dots} = +\frac{\psi_x}{\psi_y},$$

which it defines, express the rate at which one "good" will be indifferently exchanged for another. This, however, is nothing but the price of one good relatively to another. Hence the ratios

$$\psi_x : \psi_y : \dots = p_x : p_y : \dots$$

are the price ratios  $p$  relative to a fixed good as base. Be it observed, however, that these are not in general the actual market prices, but the prices which the given individual would himself establish. Moreover, the actual terms like  $F'\psi_x = I_x$  are the rates of increase of  $I$  on the marginal amounts  $dx$ , and so forth; that is, they are the marginal ophelimities or utilities, which are so fundamental in economics.

Now conversely, one may estimate the price ratios of indifferent exchanges and may thus set up a differential equation

$$\varphi_x dx + \varphi_y dy + \dots = 0$$

of which the integral would be  $I$  or a function of  $I$ . This, however, implies that such an integral exists. The discussion of the existence of this integral was suggested as a necessity by Volterra in commenting on the Italian edition of the manual.†

\* The functional symbol  $F$  may be taken more or less arbitrarily according to the scale on which  $I$  is estimated; for although the value of  $I$  may be distinguished from a greater or a smaller value, the estimation of  $I$  is hardly quantitative.

† The same suggestion was made already in 1892 by Fisher, loc. cit., p. 88, on mathematical grounds; but I believe that considerations indicated by Gibbs have since led him to abandon the suggestion.

The author now gives a careful treatment of the question, which is necessarily somewhat complicated by the every-day fact that the consumption of certain amounts,  $dx$ ,  $dy$ ,  $\dots$  of the various goods may yield very different increments of pleasure according to the order in which they are consumed—not many would care to reverse the order of their dinner menu. (Pareto, page 251).

Now this reference to a menu is exceedingly liable to cause confusion; for the order of consumption which is in question as regards the integrability of the above differential is merely a conceptual order or curve in the space of the goods  $x$ ,  $y$ ,  $\dots$ , whereas the order of consumption which is involved in the menu is a time order which either has nothing to do with the other order or has some very fundamental and subtle and neglected relation to the whole question of utility. Suppose that we could integrate the equation and thus obtain an index of pleasure; this index ought, it would seem, to apply either at a given instant (and to contain the time as a parameter, owing to the fact that at different times the price ratios and integrating factor are different), or it ought to apply to an interval of time such as a day or a year during which conditions were supposed to remain the same (and the time would then not enter at all). Fisher in his memoir distinctly takes the latter point of view, and indicates its advantages in smoothing out capricious elements, whims instead of well balanced desires. This is the time-average way of obtaining definite results. But the homo oeconomicus may well be considered an idealized averaged individual free from caprice. This is the ensemble-average way of getting definiteness. Pareto seems to us to take this point of view. (Students of statistical mechanics are acquainted with the two types of average and with their essential identity for many problems.) But in neither case does the time enter into the determination of the equilibrium or of the conceptual path followed to attain the point of maximum pleasure—unless, indeed, the whole analogy with mechanics and thermodynamics is to break down in a vital point. We must remember that in statics the time element is ignored and that in thermodynamics changes are supposed to take place infinitely slowly. The difference in the increase of pleasure which a person would experience in eating his dinner in inverse as opposed to direct order cannot, we believe, be logically taken into account, and we are inclined to think

that the whole discussion of the order of integration, a point that naturally occurs to the mathematician, can be thrown out of court by the economist. For on the ensemble-average basis with an instantaneous comparison of the desirability of increments of different goods, there is no period of time during which they can be consumed in temporal order, the "consumption" is mere conceptual appropriation; whereas on the time-average basis the consumer is free to consume in the temporal order he chooses. The author in places comes pretty near saying this himself; he might have decided to say it if he had observed that to eat one's dinner in the proper order with too little leisure or with too lengthy waits between the courses would be quite as bad as to have to eat it in the wrong order, for we suppose he would hardly desire to include the intervals of time themselves as consumed goods so as still to have different orders.

We have spoken of the exchange ratios or price ratios which the given individual would regard as fitting his desires. These are not in general the market prices, or proportional to them. Hence the homo oeconomicus will avail himself of this divergence to make exchanges which will increase his value of  $I$ , and he will pursue these exchanges until his private exchange ratios become identical with those of the market. He will thus come into equilibrium with the market with those particular amounts  $x, y, \dots$  of goods which will maximize  $I$  subject to the imposed constraints. For example, if he starts with the amounts  $x_0, y_0, \dots$  and if the market prices are constant,\* his exchanges must satisfy the equation

$$(x - x_0)p_x + (y - y_0)p_y + \dots = 0,$$

and his price ratios must satisfy the  $n - 1$  equations

$$\varphi_x : \varphi_y : \dots = p_x : p_y : \dots$$

Thus there are just  $n$  equations wherefrom to determine the final or equilibrium values of the  $n$  quantities  $x, y, \dots$ . The problem is determinate.

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\* Although in the last analysis it is the desires of all which regulate the prices in the market, it is permissible as a first approximation to many phenomena to regard these prices as constant for any one individual; his purchases and sales are insufficient materially to change the market quotations.

This simple problem in the determination of a point of equilibrium is typical of the various more complicated problems of the same sort. The analysis really amounts to formulating the equations, counting the number of unknowns, and checking it with the number of equations. To many modern mathematicians the fear that the equations might be either redundant or incompatible would probably be so strong as to deter them from seeing much of value in the analysis. But it should be remembered that not so very long ago the method of counting constants was widely used in pure mathematics even though the science was then much more highly developed toward arithmetic equations than is now the case with economics. Moreover, in a physical science the question of rigor is very different from that in mathematics; to be ultra-rigorous mathematically may be to be infra-rigorous physically. To throw out Gibbs's phase rule because its proof, being essentially a count of constants, is no proof at all would be equally good mathematics and equally bad physics. On the other hand, setting up exact mathematical relations, compatible and uniquely soluble, over the whole range of variation of the variables might be a wonderful mathematical tour de force while being viciously misleading physics in the neighborhood of certain critical points where a slight change of the variables introduces such wide variations in the functions as to make the problem just as indeterminate physically as it has become determinate mathematically.

It would not do, however, to give the impression that the use of analysis in economics is restricted to counting constants. It is necessary to study the general properties of the surfaces of indifference, and this leads to the quadratic differential form for the second differential  $d^2I$ . The law of demand and supply, in the simple case of a single individual in relation to the whole market, requires partial derivatives and transformations among them which are well comparable to those required in thermodynamic theory. If space permitted, we should be glad to stop at this point and sketch the proof of the law that the demand decreases as the price rises. It would perhaps be still more indicative of the usefulness of mathematics if we could run through the author's analysis of Marshall's hypothesis that for small variations of price and of quantities the marginal utility of money could be treated as constant; for Pareto shows by an extended series of reductions ad absurda,

which apparently would be difficult to establish without some use of mathematics, that the hypothesis is not tenable. Further on, in the discussion of production, which is probably the most complicated in the whole appendix, it becomes necessary to maximize integrals, and the author is therefore led to introduce into his analysis the ordinary algorithms of the calculus of variations.

It is thus seen that it is not merely the mathematics of a first course in calculus which may be used to advantage in economic theory. Interesting as it would be to pursue these matters in detail, we feel that we must come back to the main text and offer some short account of it. In the first chapter on general principles the author first analyzes the different aims which may lie behind the study of political economy and carefully delimits himself to the single one of developing theoretical economics. He then turns his attention to discussing the relation of a theory to the actual phenomena of which it is the theory. Although he is writing with his own subject in view, this introductory essay is conceived and carried out in such a broad spirit that it may serve almost equally well as a sort of prolegomenon to the study of any scientific theory, and as such it can be recommended for general reading.

Chapter II, an introduction to social science, is delightful reading and forms a sort of amplification of the preceding chapter upon the social and moral side. Honest and manly in its views, pithy and pungent in its epigram, it calls to mind the direct diction and unrelenting exposition of W. G. Sumner's *Folkways*. Pareto, however, did not make use of, and perhaps did not have at hand, the mass of detailed references to folkways which Sumner adduced to drive in and rivet in his opinions. For this reason those who are especially interested in this chapter should supplement their study of it by reading the *Folkways*. Indeed if economics is a misty region under a pall of prejudices, where fierce war is often waged around the banners of certain words—trusts, protection, supply and demand, single tax—of the meaning of which the combatants have not always a very clear perception, even more is this true of social science, and one who would really attain a free and independent judgment in this field has to exercise the greatest care. One of the most interesting observations of the author is (page 122) that there is an antagonism between

the conditions of action and of knowledge, that faith alone drives men to action, and that it is not for the good of society that very many persons should be occupied with the scientific study of social questions.\*

It is the next four chapters, *Notion générale de l'équilibre économique, Les goûts, Les obstacles, L'équilibre économique*, which contain for the most part that portion of the text which is supplemented by the mathematical appendix, and which is most scientific in its development. It is this part which seems so different from the older treatises on economics, not only on account of the generous use of curves as in a number of modern texts, but particularly because of the great care with which the problems arising are divided into separate sub-problems and the exhaustive manner in which the various sub-cases are treated. We have tried to give an idea of economic equilibrium in a simple case; for any idea of the multifarious distinctions which arise in the presence of various degrees of free competition and monopoly we must refer to the text. But to show how conscientious the author is in exhibiting real as well as theoretical phenomena, we will cite an instance or two. It is a statement often found in texts on economics that the more we have of a thing, the less valuable to us are equicrescent increments of it; that is, the marginal utility of anything decreases with an increasing amount of it. And many an author states this as a general law. Pareto, however, makes clear the fact that there are exceptions. It does no harm to amplify the matter even further, to point out that in this life our desires consist of a lot of little ones and a few big ones, that to satisfy the latter large increments of certain goods (say, of money) are necessary, and that therefore the curve of marginal utility of money consists in reality of long slow descents (in the intervals where the little desires are better and better satisfied, but the satisfaction of one of the large ones seems hopelessly remote) interrupted by sharp ascents (in the intervals where the fulfilment of one of the greater desires seems imminent). In another place the author, when discussing the proposition that the returns on

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\* A footnote appended to these remarks may be of interest. Par exemple le livre de M. Roosevelt, *American Ideals*, pourra peut-être servir à pousser à l'action les citoyens des Etats-Unis, mais assurément il n'ajoute rien à nos connaissances, et sa valeur scientifique est très voisine de zéro. Pareto never minces matters in the expression of his opinion, and to a careless or prejudiced reader might therefore appear unjust or bitter when he meant to be only frank.

capital differently invested ought in a state of equilibrium to be the same, cites some very interesting statistics concerning Belgian corporations. It appears that from 1873 to 1887 there were founded 1,088 companies with a paid-in capital of 1,493 millions of francs. Up to 1901 the sum of 515 millions had been paid back under liquidation, and the remaining companies were earning 56 millions a year. This is only 5.7 per cent. on the 978 millions remaining in the original investment. Although the figures are only rough, it is certain that the return on the 1,088 enterprises could not have much exceeded the current rate of interest. In this connection let us remark that some student statistician looking for a thesis might consider the problem of calculating as accurately as possible the rate of return on capital variously invested in this country.

Chapter VII is on various topics connected with the population—the distribution of wealth, relations between economic conditions and the population, malthusianism, stability and selection, and so on. In the next chapter the author takes up capital, and treats savings, interest, exchanges, international commerce, bimetalism, and the like. His definition of capital appears to us considerably less definite than that of Irving Fisher in his *Capital and Income and Rate of Interest*.\* Indeed it may be safe to hazard the guess that Pareto, who seems to esteem Fisher's work very highly, might have made considerable use of it if only it had appeared early enough to have been easily available for the *Manuel*. It goes without saying that when treating such subjects as population and capital, the author is merging into the realms of applied economics where the argument has to run somewhat differently from its previous course.

In the final chapter, *Le phénomène économique concret*, we have, among many other things, a delightful essay on present conditions and tendencies in our economic or social life. To any one who is perplexed at our rapid social evolution, who is confused in his ideas of trusts or of labor unions, who is bewildered at the way in which the rising costs of government are bleeding the bourgeoisie for the support of the lower classes, to any one who wonders where we are and whither we are going, no better reading can be suggested than this chapter of Pareto's; it is short enough to keep the attention, virile

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\* Reviewed in this BULLETIN, vol. 15, p. 169.

enough to claim the admiration, of the most nonchalant reader. Much as the author's scientific economics appeals to us, we must say that his sturdy vision into society and social evolution are even more to our liking; and there can be little doubt that for many of us, mathematicians or otherwise, the most inspired portions of Pareto's *Manuel* will be the sociology of Chapters II and IX.

In closing this review it may be only proper to apologize a bit for its tardiness. At the same time we may perhaps point out that it is now while we are in the midst of modifying constitutions, whether actually or under various disguises such as preferential primaries or judicial rule of reason, while we are attacking or about to attack the formerly sacred institution of private property, under many a pretext other than the real one that might makes right, now while we in America are perhaps more than ever conscious of our present or impending social evolution is the time when we may derive the most benefit from having our attention called to Pareto's observations on the ills that beset us. As for the mathematical economics, that is still good, and still will be good at any time when we may settle down to study it.

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#### NOTES.

THE programme of the Fifth international congress of mathematicians at Cambridge, England, begins on Wednesday evening, August 21, with a reception of the members in the Hall of St. John's College. The opening meeting will be held on Thursday morning and the afternoon will be devoted to organisation, election of officers, and two of the special lectures. On the following days sectional meetings will be held in the morning and lectures will be delivered in the afternoon. The congress will divide into four sections: I. Arithmetic, algebra, analysis; II. Geometry; III. Mechanics, physical mathematics, applications; IV. Philosophical, historical, and didactic questions. In connection with Section IV three discussions inaugurated by the International commission on the teaching of mathematics will be held. The sections will be further subdivided as the number of papers may require.