grated by Sturm's method as three distinct problems, each being obtained by a change of variables. Although the Legendrian notation is practically adhered to, the respective substitutions are equivalent to

$$(27) \qquad 1 - k^2 \sin^2 \phi = x^2, \quad \cos^2 \phi = x^2, \quad \sin^2 \phi = x^2.$$

The respective values of $f(x)$ are equivalent to

$$(28) \quad (1 - x^2)(x^2 - k'^2), \; (1 - x^2)(k'^2 + k^2x^2), \; (1 - x^2)(1 - k^2x^2).$$

Schröter's values of F are deduced for each of the three cases independently, by expressing the sufficient conditions for the vanishing of coefficients in his equation corresponding to (12) above. His results are obtainable at once by using his respective values of $f(x)$ from (28) in (25) and (24), and this affords a concrete illustration of the theorem which these two equations express.

UNIVERSITY OF PENNSYLVANIA,
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A PROPERTY OF A SPECIAL LINEAR SUBSTITUTION.

BY PROFESSOR F. R. SHARPE.

LET ξ_i denote homogeneous line coordinates * which satisfy the quadratic identity

$$(1) \qquad \Sigma\xi_i^2 = 0.$$

The condition that two lines ξ, ξ' intersect is

$$(2) \qquad \Sigma\xi_i\xi_i' = 0.$$

The equation of a linear complex is of the form

$$(3) \qquad \Sigma a_i\xi_i = 0.$$

It is clear from the above equations that, when

$$(4) \qquad \Sigma a_i^2 = 0,$$

the lines of (3) all intersect the line a; (3) is then called a special complex.

* Jessop: Treatise on the line complex, Arts 9, 15–20.

If

$$\lambda = -\frac{2\Sigma a_i x_i}{\Sigma a_i^2},$$

then

$$\Sigma(x_i + \lambda a_i)^2 = \Sigma x_i^2.$$

Hence

(5) $$\Sigma x_i' \xi_i = \Sigma(2\Sigma a_i x_i \cdot a_i - \Sigma a_i^2 \cdot x_i)\xi_i = 0$$

is a special complex if

(6) $$\Sigma x_i \xi_i = 0$$

is a special complex.

The linear complex (5) may be termed the transform of (6) through (3).

Two complexes x and y are said to be in involution if

(7) $$\Sigma x_i y_i = 0.$$

When the two complexes are transformed through a

(8) $$\Sigma x' y' = (\Sigma a_i^2)^2 \cdot \Sigma x_i y_i = 0.$$

Hence the transforms are also in involution. Consider the transformations

(A) $$x_i' = 2\Sigma a_i x_i \cdot a_i - \Sigma a_i^2 \cdot x_i,$$

(B) $$x_i'' = 2\Sigma b_i x_i' \cdot b_i - \Sigma b_i^2 \cdot x_i',$$

which transform x through the complexes a and b respectively, and

(A') $$x_i''' = 2\Sigma(sa_i + tb_i)x_i \cdot (sa_i + tb_i) - \Sigma(sa_i + tb_i)^2 \cdot x_i,$$

(B') $$x_i^{\mathrm{IV}} = 2\Sigma(s'a_i + t'b_i)x_i''' \cdot (s'a_i + t'b_i) - \Sigma(s'a_i + t'b_i)^2 \cdot x_i''',$$

which transform x through two complexes of the pencil defined by a and b. These transformations have the property that if A, B, and A' are given, then B' can be uniquely determined so that either

(9) $$AB = A'B'$$

or

(9)* $$AB = B'A'.$$

This can be proved very simply as follows. The coefficient of x_j in AB is

$$4a_j b_i \Sigma a_i b_i - 2a_j a_i \Sigma b_i^2 - 2b_j b_i \Sigma a_i^2.$$

If (9) is satisfied, the corresponding coefficients must have a common ratio ρ, so that

$$(10) \qquad x_i'' \equiv \rho x_i^{\mathrm{IV}}.$$

Hence

$$\rho = \frac{\text{coefficient of } x_j \text{ in } x_i''}{\text{coefficient of } x_j \text{ in } x_i^{\mathrm{IV}}} = \frac{\text{coefficient of } x_i \text{ in } x_j''}{\text{coefficient of } x_i \text{ in } x_j^{\mathrm{IV}}}.$$

Subtracting the numerators and denominators, we find after a little reduction

$$(11) \qquad \rho = \frac{\Sigma a_i b_i}{(st' - s't)\Sigma(sa_i + tb_i,\ s'a_i + t'b_i)}.$$

Similarly by addition we find

$$(12) \qquad \rho = \frac{1}{(st' - s't)^2}.$$

From (11) and (12) it follows that

$$(13) \qquad \Sigma(sa_i + tb_i,\ s'a_i + t'b_i) = (st' - s't)\Sigma a_i b_i,$$

that is

$$(13)^* \qquad ss'\Sigma a_i^2 + 2s't\Sigma a_i b_i + tt'\Sigma b_i^2 = 0.$$

Solving, we have

$$(14) \qquad \frac{s'}{t'} = \frac{-t\Sigma b_i^2}{s\Sigma a_i^2 + 2t\Sigma a_i b_i}.$$

If we take (9)* instead of (9), then (13) becomes

$$(15) \qquad \Sigma(sa_i + tb_i,\ s'a' + t'b_i) = (s't - st')\Sigma a_i b_i$$

and (14) becomes

$$(16) \qquad \frac{s'}{t'} = \frac{t\Sigma b_i^2 + 2s\Sigma a_i b_i}{-s\Sigma a_i^2}.$$

The equations (13) and (15) are both included in

$$(17) \qquad \{\Sigma(sa_i + tb_i,\ s'a_i + t'b_i)\}^2 = (st' - s't)^2\{\Sigma^2 a_i b_i\}^2,$$

but

$$(18) \qquad \begin{aligned} \{\Sigma(sa_i + tb_i,\ s'a_i + t'b_i)\}^2 - \Sigma(sa_i + tb_i)^2 \cdot \{\Sigma(s'a_i + t'b_i)^2 \\ = (st' - s't)^2\{(\Sigma a_i b_i)^2 - \Sigma a_i^2 \cdot \Sigma b_i^2\}. \end{aligned}$$

Hence (17) may be written in the form

$$(19) \qquad \frac{\{\Sigma(sa_i + tb_i, s'a_i + t'b_i)\}^2}{\Sigma(sa_i + tb_i)^2 \cdot \Sigma(s'a_i + t'b_i)^2} = \frac{\{\Sigma a_i b_i\}^2}{\Sigma a_i^2 \cdot \Sigma b_i^2}.$$

It is important to notice that the roots of the quadratic (19) in s'/t' are rationally separable and are given by (14) and (16), which are the solutions of (9) and (9)* respectively.

It remains to show that

$$(20) \qquad \rho = \frac{\text{coefficient of } x_i \text{ in } x_i''}{\text{coefficient of } x_i \text{ in } x_i^{\mathrm{IV}}}.$$

The coefficients are similar to those which were used in (11) and (12), but have an extra term

$$(21) \qquad \Sigma a_i^2 \cdot \Sigma b_i^2 \quad \text{and} \quad \Sigma(sa_i + tb_i)^2 \cdot \Sigma(s'a_i + t'b_i)^2$$

respectively. From (17) and (18) it follows that the terms (21) are also in the ratio 1 to $(st' - s't)^2$. This completes the proof of the property. The property may be expressed in terms of the geometry of the sphere.*

If we regard the x_i as point coordinates in hyperspace, then (5) shows that the quadratic variety

$$\Sigma x_i^2 = 0$$

is invariant under the transformation and (8) shows that polarity is preserved. If in (13)* we put

$$s = 0, \quad t' = 0,$$

we find that the condition that A and B are commutative is

$$\Sigma a_i b_i = 0.$$

Hence it follows that A and B are commutative when a and b are in involution.

CORNELL UNIVERSITY,
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* P. F. Smith, *Transactions*, vol. 1 (1900), p. 378, footnote. The statement in the footnote overlooks the distinction between our equations (14), (16).