ON FREDHOLM'S EQUATION.

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THE formulas which Fredholm has given* for the solution of the integral equation

$$(1) \qquad \phi(x) + \lambda \int_0^1 \kappa(x, s)\,\phi(s)\,ds = \psi(x),$$

in which λ is a given constant, $\kappa(x, y)$ and $\psi(x)$ are given functions, while $\phi(x)$ is to be determined, are well known. Fredholm remarks that the theory of this equation may be considered as a limiting case of the theory of a set of linear algebraic equations, but in his published demonstration he makes no use of this remark and contents himself with a verification of his formulas. Hilbert† and Plemelj,‡ on the other hand, have obtained Fredholm's results by considering an infinite set of linear equations. More recently, Goursat § and Lebesgue || have shown that for a kernel $\kappa(x, y)$ having the form

$$\kappa(x, y) = X_1(x)\,Y_1(y) + X_2(x)\,Y_2(y) + \cdots + X_n(x)\,Y_n(y),$$

one can easily obtain Fredholm's formulas. In the present note, I should like to show how the consideration of equation (1) leads directly to Fredholm's expressions.

If we set

$$(2) \qquad \phi(x) = \psi(x) - \rho(x),$$

* "Sur une nouvelle méthode pour la résolution du problème de Dirichlet," *Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar*, Stockholm, vol. 57, p. 39 (1900). "Sur une classe de transformations rationnelles," *Comptes rendus*, vol. 134, p. 219 (1902). "Sur une classe d'équations fonctionnelles," *Acta Mathematica*, vol. 27, p. 365 (1903).

† "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Göttinger Nachrichten*, p. 49 (1904).

‡ "Zur Theorie der Fredholmschen Funktionalgleichung," *Monatshefte für Mathematik und Physik*, vol. 15, p. 93 (1904).

§ "Sur un cas élémentaire de l'équation de Fredholm," *Bulletin de la Société mathématique de France*, vol. 35, p. 163 (1907).

|| "Sur la méthode de M. Goursat pour la résolution de l'équation de Fredholm," *Bulletin de la Société mathématique de France*, vol. 36, p. 3 (1908).

equation (1) becomes

$$\rho(x) + \lambda \int_0^1 \kappa(x, s)\rho(s)\,ds = \lambda \int_0^1 \kappa(x, s)\psi(s)\,ds. \tag{3}$$

Before attempting to solve this equation in all its generality let us first examine the special case in which $\psi(x)$ is identically equal to zero. The resulting equation

$$\rho(x) + \lambda \int_0^1 \kappa(x, s)\rho(s)ds = 0 \tag{4}$$

is obviously satisfied by taking $\rho(x)$ identically equal to zero. But the analogy that exists between this equation and a set of homogeneous linear algebraic equations makes it natural to attempt to determine the expression whose vanishing is necessary in order that the equation may admit of a solution not identically equal to zero.

From equation (4) we get

$$\rho(x) = -\lambda \int_0^1 \kappa(x, s_1)\rho(s_1)ds_1. \tag{5}$$

If we add to each side of this equation the expression

$$\lambda\rho(x)\int_0^1 \kappa(s_1, s_1)ds_1,$$

the integrand on the right will take the form of a determinant and we get

$$\rho(x)\left[1 + \lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] = \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_1) \\ \rho(s_1) & \kappa(s_1, s_1) \end{vmatrix} ds_1 \tag{6}$$

$$= \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_2) \\ \rho(s_2) & \kappa(s_2, s_2) \end{vmatrix} ds_2. \tag{7}$$

In the second member of equation (7) let us replace $\rho(x)$, $\rho(s_2)$ by the values given by equation (5); we get

$$\begin{aligned} &\rho(x)\left[1 + \lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] \\ &\qquad = -\lambda^2\int_0^1\int_0^1 \rho(s_1)\begin{vmatrix} \kappa(x, s_1) & \kappa(x, s_2) \\ \kappa(s_2, s_1) & \kappa(s_2, s_2) \end{vmatrix} ds_1 ds_2. \end{aligned} \tag{8}$$

An interchange of the subscripts 1, 2 in the expression on the right of this equation will not change its value; we may therefore write

$$
(9)\quad \begin{aligned}\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] = \frac{\lambda^2}{2}\int_0^1\int_0^1\Bigg[-\rho(s_1)&\begin{vmatrix}\kappa(x, s_1) & \kappa(x, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix}\\ -\rho(s_2)&\begin{vmatrix}\kappa(x, s_2) & \kappa(x, s_1)\\ \kappa(s_1, s_2) & \kappa(s_1, s_1)\end{vmatrix}\Bigg] ds_1 ds_2.\end{aligned}
$$

On adding

$$
\frac{\lambda^2}{2}\rho(x)\int_0^1\int_0^1\begin{vmatrix}\kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix} ds_1 ds_2
$$

to each side of this equation, the integrand on the right will take the form of a determinant and we get

$$
(10)\quad \begin{aligned}\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1 + \frac{\lambda^2}{2}\int_0^1\int_0^1\begin{vmatrix}\kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix} ds_1 ds_2\right]\\ = \frac{\lambda^2}{2}\int_0^1\int_0^1\begin{vmatrix}\rho(x) & \kappa(x, s_1) & \kappa(x, s_2)\\ \rho(s_1) & \kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \rho(s_2) & \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix} ds_1 ds_2.\end{aligned}
$$

Equations (6) and (10) suggest at once the general formula

$$
(11)\quad \begin{aligned}\rho(x)\left[1+\sum_1^n\frac{\lambda^p}{p!}\int_0^1\int_0^1\cdots\int_0^1\kappa\begin{pmatrix}s_1, s_2, \cdots, s_p\\ s_1, s_2, \cdots, s_p\end{pmatrix}ds_1 ds_2\cdots ds_p\right]\\ = \frac{\lambda^n}{n!}\int_0^1\int_0^1\cdots\int_0^1\begin{vmatrix}\rho(x) & \kappa(x, s_1) & \cdots & \kappa(x, s_n)\\ \rho(s_1) & \kappa(s_1, s_1) & \cdots & \kappa(s_1, s_n)\\ \vdots & \vdots & & \vdots\\ \rho(s_n) & \kappa(s_n, s_1) & \cdots & \kappa(s_n, s_n)\end{vmatrix} ds_1 ds_2\cdots ds_n,\end{aligned}
$$

in which

$$
\kappa\begin{pmatrix}s_1, s_2, \cdots, s_p\\ s_1, s_2, \cdots, s_p\end{pmatrix}
$$

is used to denote the determinant of the pth order in which the element in the ith row and jth column is $\kappa(s_i, s_j)$.

It is not hard to establish equation (11) by induction. Moreover, by using Hadamard's theorem * that the modulus of a determinant of the nth order is less than $M^n\sqrt{n^n}$, M being a number greater than the modulus of every element, it is possible to show that as n increases without limit the right side of equation (11) approaches zero as a limit. Accordingly it follows that in order that equation (4) may have a solution not identically zero it is necessary that

$$D(\lambda) = 0,$$

where

$$(12)\quad D(\lambda)=1+\sum_1^\infty \frac{\lambda^p}{p!}\int_0^1\int_0^1\cdots\int_0^1 \kappa\begin{pmatrix} s_1, & s_2, & \cdots, & s_p \\ s_1, & s_2, & \cdots, & s_p\end{pmatrix} ds_1 ds_2\cdots ds_p.$$

As is well known, the convergence of (12) for every value of λ can be established by means of Hadamard's theorem.

Let us now return to equation (3). Guided by the analogy which exists between this equation and a set of non-homogeneous linear equations we may assume that $\rho(x)$ is equal to a fraction of which $D(\lambda)$ is the denominator. Accordingly, if we set

$$\rho(x) = \frac{\sum_0^\infty \lambda^n A_n(x)}{D(\lambda)},$$

and if we substitute in equation (3) we shall find without much difficulty the values of the functions $A_n(x)$; we shall thus obtain Fredholm's formulas.

We can reach the same result more simply, however, by treating equation (3) as we have treated equation (4). From equation (3) we get

$$(13)\quad \rho(x) = -\lambda\int_0^1 \kappa(x, s_1)\rho(s_1)ds_1 + \lambda\int_0^1 \kappa(x, s)\psi(s)ds.$$

As before, we add

$$\lambda\rho(x)\int_0^1 \kappa(s_1, s_1)ds_1$$

to each side of this equation and we obtain an equation analogous to equation (7)

* "Résolution d'une question relative aux déterminants." *Bulletin des Sciences Mathématiques*, 2d series, vol. 17, p. 240 (1893). Cf. W. Wirtinger, "Sur le théorème de M. Hadamard relatif aux déterminants," ibid., 2d series, vol. 31, p. 175 (1907).

$$
(14)\quad \rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] = \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_2) \\ \rho(s_2) & \kappa(s_2, s_2) \end{vmatrix} ds_2 + \lambda\int_0^1 \kappa(x, s)\psi(s)ds.
$$

In the second member of this equation let us replace $\rho(x)$, $\rho(s_2)$ by the values given by equation (13). We thus get an equation analogous to equation (8)

$$
(15)\quad \begin{aligned} \rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] &= -\lambda^2\int_0^1\int_0^1 \rho(s_1)\begin{vmatrix} \kappa(x, s_1) & \kappa(x, s_2) \\ \kappa(s_2, s_1) & \kappa(s_2, s_2) \end{vmatrix} ds_1 ds_2 \\ &\quad + \lambda\int_0^1 \kappa(x, s)\psi(s)ds \\ &\quad + \lambda^2\int_0^1\int_0^1 \begin{vmatrix} \kappa(x, s) & \kappa(x, s_1) \\ \kappa(s_1, s) & \kappa(s_1, s_1) \end{vmatrix} \psi(s)ds_1 ds. \end{aligned}
$$

If we treat this equation as we treated equation (8), it is clear that we shall obtain an equation like equation (10) but differing from it by the presence of the two additional terms

$$
\lambda\int_0^1 \kappa(x, s)\psi(s)ds + \lambda^2\int_0^1\int_0^1 \begin{vmatrix} \kappa(x, s) & \kappa(x, s_1) \\ \kappa(s_1, s) & \kappa(s_1, s_1) \end{vmatrix} \psi(s)ds_1 ds
$$

on the right. Continuing in this way it is not hard to see that we shall get instead of equation (11) an equation differing from it by the presence on the right of the expression

$$
(16)\quad \int_0^1 \psi(s)F_n(x, s, \lambda)ds
$$

where

$$
(17)\quad \begin{aligned} F_n(x, y, \lambda) &= \lambda\kappa(x, y) \\ &\quad + \sum_1^{n-1}\frac{\lambda^{p+1}}{p\,!}\int_0^1\int_0^1\cdots\int_0^1 \kappa\begin{pmatrix} x, & s_1, & \cdots, & s_p \\ y, & s_1, & \cdots, & s_p \end{pmatrix} ds_1 ds_2\cdots ds_p. \end{aligned}
$$

If, in this new equation, analogous to (11), we allow n to increase without limit we obtain

$$(18)\qquad \rho(x)D(\lambda) = \int_0^1 \psi(s)F(x, s, \lambda)ds,$$

where $F(x, y, \lambda)$ denotes the limit to which $F_n(x, y, \lambda)$ tends,

$$F(x, y, \lambda) = \lambda\kappa(x, y)$$

$$(19)\qquad + \sum_1^\infty \frac{\lambda^{p+1}}{p!} \int_0^1 \int_0^1 \cdots \int_0^1 \kappa\begin{pmatrix} x, & s_1, & \cdots, & s_p \\ y, & s_1, & \cdots, & s_p \end{pmatrix} ds_1\, ds_2 \cdots ds_p.$$

The convergence of this expression can be established by Hadamard's theorem.

If, in equation (18), we replace $\rho(x)$ by its value taken from equation (2), we get Fredholm's solution of equation (1)

$$(20)\qquad \phi(x) = \psi(x) - \frac{1}{D(\lambda)} \int_0^1 F(x, s, \lambda)\psi(s)\, ds.$$

It is to be observed that the above demonstration establishes the uniqueness of the solution for every value of λ for which $D(\lambda) \neq 0$.

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ENGINEERING education, having passed through an infancy somewhat of the "ugly duckling" type in the academic family, and then an adolescence of growth too rapid for either garments or comfort, seems now to have arrived at a stage of maturity in which it recognizes the necessity for heart-searching as to its own real character and habits of life. The demands upon its energies from all quarters are heavy and insistent. It must maintain its vigor and efficiency by ridding itself of every