THE SECOND REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE second regular meeting of the Southwestern Section of the Society was held at the University of Kansas, Lawrence, Kansas, on Saturday, November 28, 1908. About fifty persons, including the following seventeen members of the Society, were present:

Professor W. C. Brenke, Miss Julia T. Colpitts, Professor E. C. Colpitts, Professor E. W. Davis, Dr. Otto Dunkel, Professor E. P. R. Duval, Professor C. C. Engberg, Professor A. B. Frizell, Mr. G. W. Hartwell, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. W. A. Luby, Professor H. B. Newson, Dr. Mary W. Newson, Professor W. H. Roever, Professor H. E. Slaught, Professor Paul Wernicke.

The morning session was opened at 10 A. M. and the afternoon session at 2 P. M., Professor Newson presiding. The reading of papers was followed by a business meeting, at which Columbia, Mo., was fixed as the next place of meeting, and Professors Davis (chairman), Roever, and Kellogg (secretary) were elected as program committee for the ensuing year.

The following papers were presented:

(1) Professor J. N. VAN DER VRIES: "The steinerians of quartic surfaces."

(2) Professor G. A. MILLER: "On the groups generated by two operators satisfying the condition $s_1s_2 = s_2^{-2}s_1^{-2}$."

(3) Professor E. W. DAVIS: "The imaginary elements of the exponential curve."

(4) Mr. MEYER GABA: "A necessary condition for the Cremona transformation of curves" (preliminary communication).

(5) Professor W. M. ROEVER: "Optical interpretation of some problems in statics."

(6) Professor W. C. BRENKE: "Transformation of series by means of functions admitting a recurrent relation."

(7) Professor A. B. FRIZELL: "Some sets whose cardinal is the second transfinite number."

(8) Professor PAUL WERNICKE: "Note on linkages."

(9) Professor A. B. FRIZELL: "A theorem on operation groups."

(10) Professor E. R. HEDRICK: "Integrals independent of the path in a complex field."

(11) Mr. G. W. HARTWELL: "Fields of force invariant under projective transformations."

(12) Professor PAUL WERNICKE: "Note on curvatures."

Professor Van der Vries and Mr. Gaba were introduced by Professor Newson. Professor Miller being absent, his paper was read by title. This paper appeared in the January BULLETIN. Abstracts of the other papers follow, numbered to correspond with the above list.

1. Professor Van der Vries considered the steinerian of the developable quartic both geometrically and analytically, and finds that it represents all of space. The quartic

$$ax^2s^2 + by^2z^2 + xyzs = 0$$

is examined analytically and the steinerian is found to be each of the four uninodal planes of the surface seven times and another quartic surface having all the singularities of the original quartic. The steinerian of the general quartic surface being of degree 32 is too difficult to admit of analytical methods. Surfaces having particular singularities are considered and the corresponding points of the steinerian located. The paper closes with a few general statements and proofs of properties of hessians and the corresponding properties of the steinerians.

3. Using the method of representing imaginaries which Professor Davis has previously presented to the Society, the periodicity of the exponential function as well as the peculiarity at infinity are interpreted geometrically.

4. Every Cremona transformation can be reduced to a series of quadratic Cremona transformations and collineations. A well-known necessary condition that a curve be transformable into another by a quadratic (and consequently by the general) Cremona transformation is that the genus of the two curves be the same. In Mr. Gaba's paper, a definition of complete rth adjoint, variable rth adjoint, and complement is given. A second necessary condition for the transformability of one curve into another is then derived., viz., that both curves have rth

variable adjoints of the same genus and the same degree of freedom, and that both curves lack the same variable sth adjoints.

5. Consider an apparatus consisting of a smooth wire and two small rings, P_1 and P_2, not on the wire, all rigidly attached to a frame. The wire is in the form of any curve, and on it a small weightless ring R is capable of sliding without friction. A weightless and perfectly flexible string with a weight at one end, after passing through the rings P_2 and R, has its other end attached to P_1. The string slides through the rings P_2 and R without friction. The ring R, under the action of the forces which act upon it, will be in equilibrium at certain points P of the wire. If now the rings P_1 and P_2 be replaced by the eye of an observer and a point source of light respectively, the observer will see images (*i. e.*, actual brilliant points *) of the light at certain points Q of the wire. The first theorem of Professor Roever's paper is that the points P and Q are identical.

Let us now think of a weightless and perfectly flexible string with both of its ends attached at fixed points P_1 and P_2. On the string a small *heavy* ring R is capable of motion without friction. Consider this apparatus as situated in any field of force for which a force function exists. The ring R, under the action of the forces which act upon it, will be in equilibrium at certain points P of the field. If now the points P_1 and P_2 be replaced by the eye of an observer and a point source of light respectively, the observer will see at P an image of the light (*i. e.*, an actual brilliant point) in the equipotential surface which passes through P. If no force function exists, the point P will be a virtual extra brilliant point of the line of force which passes through P. If in particular the points P_1 and P_2 coincide, the apparatus becomes a plumb line. Then the point P is the image which the observer sees of his eye in the equipotential surface which passes through P.

An apparatus was shown which illustrates the first theorem, and an apparatus was described which illustrates the second theorem. In the second apparatus the equipotential surface is actually assumed by the surface of water which reflects light.

6. In Professor Brenke's paper the series $\sum_1^\infty u_n\psi_n(x)$ is transformed by means of an assumed recurrent relation of the

* For the definitions of the different kinds of brilliant points, see *Transactions Amer. Math. Society*, vol. 9, No. 2, pp. 245–279.

form $\phi_t\psi_n = \alpha_n\psi_{n+r} + \beta_n\psi_{n+s}$, and a theorem on the convergence of the series is stated. Application is made to the cases where $\psi_n(x)$ is one of the functions cos nx, Hermite's polynomial $U_n(x)$, Legendre's polynomial $X_n(x)$, and Bessel's function $J_n(x)$. By specializing the value of x, certain transformations and convergence tests are obtained for the series $\sum_1^\infty u_n$.

7. All permutations of a finite set of symbols may be obtained from any one of them by the enumerative process of filling the first place in all possible ways, then the second, and so on. The example of the continuum proves that this does not hold for infinite sets. Professor Frizell points out that a set whose cardinal is that of all permutations of the series 1 2 3 . . . can be obtained from this series by transposing pairs of consecutive numbers, *e. g.*, 21 43 56 . . . a property which obviously is not shared by finite sets. Transpositions introduced according to the method used by Hardy in building up a set whose cardinal is the second transfinite number $\aleph_1$ furnish an interesting set of sequences with the same cardinal.

8. If by means of non-intersecting lines (links) we connect $2v$ points (vertices) in 3-space, linking each of them to three others, we obtain a linkage with simple vertices. Such linkages are formed by the boundaries of certain maps. The proof of the map color theorem rests on showing that the links of a map's linkage may be marked by 3 indices α, β, γ so that no two bearing the same mark concur in a vertex. It is not possible, as Tait suggests, to apply such marking to the general simple vertex linkage, but it is shown in Professor Wernicke's paper that the conditions under which it can be done, cover the case of the "planar" linkage (that of the plane map).

9. Professor Frizell's second paper establishes the following proposition : Given two rules of combination of which the first is distributive relatively to the second, and a denumerable set of symbols forming a group with respect to the first rule, a semi-group with respect to the second, and a simply ordered class with reference to both, it is possible so to define extensions of the two given rules to the class of fundamental sequences of the given symbols that it shall possess the given properties with regard to the extended rules.

10. Independence of the path of integration is a characteristic property of analytic functions of a complex variable. Professor Hedrick first discusses integrals of the form $I_c = \int_c f'(z)dz$, where $f(z)$ is not analytic, and where $f'(z)$ is the value of the derivative in the direction of the path C. It results that I_c is independent of the path, that $\int_{z_0}^{z} f'(z)dz = f(z) - f(z_0)$, and that the values of $f'(z)$ on the elements of any radial field determine $f(z)$ except for a constant. The properties of $f'(z)$ on such a field are discussed.

In close connection with the preceding stands the discussion of integrals of the form $I_c = \int_c f(z)d\phi(z)$, where $f(z)$ and $\phi(z)$ are any functions of a complex variable. Such integrals, for real functions, have been considered by Kowalewski and by Ingold. Professor Hedrick shows that the integral I_c will be independent of the path, and that the formula for integration by parts holds if $f(z)$ and $\phi(z)$ together satisfy certain symmetrical conditions analogous to the Cauchy-Riemann equations, which turn out to be precisely the Beltrami forms for the surface element computed from $\phi(z)$.

11. A particle moving freely in a plane under the action of a force which depends only on the position of the particle has equations of motion of the form

$$md^2x/dt^2 = \phi(x, y), \quad md^2y/dt^2 = \psi(x, y).$$

The differential equation of the trajectories described by a particle under such conditions has the form

$$y''' = Hy''^2 + Gy'',$$

in which H and G are functions of y', ϕ, ψ, and the first partial derivatives of ϕ and ψ. This type of differential equation is invariant only under the group of collineations. In Mr. Hartwell's paper, the functions ϕ and ψ are found so that this differential equation will be invariant under the several groups of collineations classified by Lie. These functions do not exist in the case of groups of more than five parameters. The integral equations of the trajectories are readily obtained when the group under which the trajectories are invariant contains three or more parameters. The paper will appear in the *Transactions*.

12. For $n = 2, 3$, etc., the determinant

$$\begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 1 & x_1' & x_2' & \cdots & x_n' \\ 1 & x_1'' & x_2'' & \cdots & x_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_1^{(n+1)} & x_2^{(n+1)} & \cdots & x_n^{(n+1)} \end{vmatrix}$$

represents the n !-fold area, volume, etc., of the figure in n-space determined by $n + 1$ points $(x_1, x_2 \cdots, x_n)$, $(x_1', x_2', \cdots, x_n')$, $\cdots$, $(x_1^{(n+1)}, x_2^{(n+1)}, \cdots, x_n^{(n+1)})$ and by the (higher) planes connecting every two, every three of them, etc. This figure possesses $\frac{1}{2}(n + 1)\, n$ edges: $P_{01}, P_{02}, \cdots, P_{n,n+1}$, of lengths p_{01}, $p_{02} \cdots, p_{n,n+1}$, respectively; $(n + 1)\, n\, (n - 1)/3\,!$ 2-planes: P_{012}, $P_{013} \cdots, P_{n-1,n,n+1}$; $\cdots$ finally $n + 1$ $(n - 1)$-planes: $P_{0,1\ldots,n}$, $P_{0,1\ldots,n+1} \cdots, P_{1,2\ldots,n+1}$.

Another expression for the same quantity is obtained by repeated multiplication of "bases" with corresponding altitudes:

$$p_{01} \cdot p_{12} \sin(P_{01}, P_{12}) \cdot p_{23} \sin(P_{12}, P_{23}) \sin(P_{012}, P_{123})$$
$$\cdot\, p_{34} \sin(P_{23}, P_{34}) \sin(P_{123}, P_{234}) \sin(P_{0123}, P_{1234}) \cdot \cdots$$

Taking $n + 1$ "consecutive" points on a general curve in n-space, and dividing each factor of this expression by the arc element ds, it becomes $c_1^{n-1} c_2^{n-2} \cdots c_{n-1}$, c_v denoting the vth curvature, while the determinant, divided by $ds^{\frac{1}{2}(n+1)n}$, easily reduces to

$$\begin{vmatrix} x_1' & x_2' & \cdots & x_n' \\ x_1'' & x_2'' & \cdots & x_n'' \\ \vdots & \vdots & & \vdots \\ x_1^{(n+1)} & x_2^{(n+1)} & \cdots & x_n^{(n+1)} \end{vmatrix},$$

where the upper indices now refer to differentiation with respect to s. Professor Wernicke thus gives a simple and general method for obtaining expressions for the various curvatures of curves in n-space.

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Secretary of the Section.