$$\lambda = \frac{x(t_2) - x(t_1)}{t_2 - t_1}, \qquad \mu = \frac{y(t_2) - y(t_1)}{t_2 - t_1} \quad \text{when} \quad t_1 \neq t_2$$

and

$$\lambda = \frac{dx_1}{dt}, \qquad \mu = \frac{dy_1}{dt}, \qquad \text{when} \quad t_1 = t_2.$$

Define the *positive sense* along this line to be the direction of increasing $\rho$. This is uniquely defined for any pair of points in the interval $(t_0, T)$ since $\lambda$ and $\mu$ are unchanged by interchanging $t_1$ and $t_2$.

Define an angle $\alpha$ as follows:

$$\sin \alpha = \epsilon k \mu, \quad \cos \alpha = \epsilon k \lambda, \quad k = (\lambda^2 + \mu^2)^{-\frac{1}{2}}.$$

Then $\alpha$ is an infinitely many-valued function of $t_1$ and $t_2$, its values for any given pair of values of $t_1$ and $t_2$ differing by multiples of $2\pi$. If one of these values $\alpha'$ be assigned to a particular pair $t_1'$, $t_2'$, then from the possible values of $\alpha$ one and only one single valued continuous function can be chosen which takes the value $\alpha'$ at $t_1'$, $t_2'$.*

UNIVERSITY OF MISSOURI,
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# ON EULER'S $\phi$-FUNCTION.

BY PROFESSOR R. D. CARMICHAEL.

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THE object of the present note is the demonstration of certain very elementary propositions concerning Euler's $\phi$-function of a number.

I. *The relation $\phi(m) = n$, a given number, is never uniquely satisfied for any given value of $n$.* That is, there is always more than one value of $m$ for every possible value of $n$.

If any solution is $m =$ an odd number, then the given relation is satisfied by $2m$ also. Likewise, if $m$ is twice an odd number, we may show that $m/2$ will also satisfy the relation.

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* Cf., e. g., Stolz, Differential-Rechnung, vol. 2, p. 15–20.

Hence, if there is a unique solution, $m$ is a multiple of 4; say $m = 4\mu$. Now $n$ is even; say $n = 2\nu$. Then we have

$$\phi(4\mu) = 2\nu.$$

Hence

$$\phi(2\mu) = \nu.$$

Then in a manner similar to the above we may show that $\mu$ and $\nu$ are both even. By continuing the process step by step we are able to show that a unique solution cannot exist unless both $m$ and $n$ are powers of 2. It remains therefore to show that this cannot give a unique solution. Let $n = 2^\alpha$. Then

$$\phi(m) = 2^\alpha$$

is satisfied not only by $m = 2^{\alpha+1}$ but also by $m = 2^a(2^b + 1)(2^c + 1) \cdots$ in every way in which $a, b, c, \cdots$ can be so chosen that $a + b + c + \cdots = \alpha + 1$, $a \neq 0$; or $b + c \cdots = \alpha$, $a = 0$; and $2^b + 1$, $2^c + 1, \cdots$ shall be different primes. If $\alpha \geqq 3$, one such solution is always $a = \alpha - 2$, $b = 1$, $c = 2$. An examination for the smaller values of $\alpha$ shows that no unique solution exists in these cases. Hence the proposition

II. *The equation* $\phi(m) = 2^n$ *has just* $n + 2$ *solutions when* $n + 2 \leqq 33$; *but just* 33 *solutions for* $n =$ *any number from* 32 *to* 255.

An odd solution evidently requires

$$m = (2^\alpha + 1)(2^\beta + 1)(2^\gamma + 1) \cdots,$$

where $\alpha + \beta + \gamma + \cdots = n$ and each factor is prime. Since $2^x + 1$ is prime* for $x = 1, 2, 4, 8, 16$, but not for $x = 32, 64, 128$ nor any value of $x$ not of the form $2^p$, it is clear that $\phi(m) = 2^n$ has one and but one odd solution for every value of $n$ up to $n = 31$. Also twice every even value of $m$ which satisfies $\phi(m) = 2^n$ will satisfy the equation when the second member becomes $2^{n+1}$. Hence the number of solutions is increased by one when $n$ is so increased up to $n = 31$; but beyond that up to $n = 255$ the number of solutions remains constant; for there is then no solution except those given by twice each solution for the preceding value of $n$. Up to $n = 31$ it is easy to see that the number of solutions is $n + 2$; then from this point onward to $n = 255$ the number remains constant and is 33.

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*See BULLETIN, June 1906, p. 449.

It will be noticed that the first value of $2^n$ to which there corresponds no odd solution in $m$ is $2^{32}$. This is the smallest value of $\phi(m)$ known to the writer to have no odd solution in $m$.

III. COROLLARY. *The equation* $\phi(m) = 2^n$ *has (only) one odd solution when* $n \leqq 31$; *otherwise no odd solution at all up to* $n = 255$. *Also it has evidently no other odd solution except for such values of* $n$ *as make* $2^n + 1$ *prime.*

IV. *All the solutions of the equation* $\phi(m) = 4n - 2$, $n \neq 1$, *are of the form* $p^\alpha$ *and* $2p^\alpha$, *where* $p$ *is a prime of the form* $4s - 1$.

Now $m \neq 4$. Then if $m$ contains the factor 4 it is evident that the equation is not satisfied. Neither is it satisfied if $m$ contains two odd primes. Therefore the only values left are of the form $p^\alpha$ and $2p^\alpha$. Moreover $p$ must be a prime of the form $4s - 1$; for otherwise the equation is not satisfied. (There may evidently be more than one $p$ which furnishes such a solution. A case in point is $\phi(m) = 18$, which has the solutions $m = 19$, 27, 38, 54.)

V. *If* $p$ *is of the form* $4s - 1$ *and* $\phi(m) = p^\alpha (p - 1)$ *has but the two solutions* $m = p^{\alpha+1}$, $2p^{\alpha+1}$, *then the relation* $\phi(m) = 2p^\alpha (p - 1)$ *has an odd solution.* ($\alpha$ belongs to the series 0, 1, 2, $\cdots$.)

For one solution of the latter is $m = 4p^{\alpha+1}$. There is no other solution in which $m$ is a multiple of 4; for then there would correspond to that a third solution for $\phi(m) = p^\alpha (p - 1)$. But $\phi(m) = 2p^\alpha (p - 1)$, by proposition I, has more than one solution. Hence it is easy to see that it has both an odd solution and another twice that one.

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