SHORTER NOTICES.

Handbuch der Theorie der Cylinderfunktionen. Von NIELS NIELSEN, Privatdozent an der Universität Kopenhagen, Inspektor des mathematischen Unterrichts an den Gymnasien Dänemarks. Leipzig, Teubner, 1904. xii + 408 pp.

DR. NIELSEN'S treatise contains twenty-seven chapters, of which all but four are largely devoted to an exposition of his own researches. While many of the results are not new, he has given more than a score of new equations and over half as many generalizations of known theorems, also numerous new proofs. Some of these proofs are for theorems stated but not proved by Lommel, Hurwitz, Jacobi, H. F. Weber and others. In several cases his proofs are intended to supersede less rigorous proofs by Sonin, Mehler and Ermakoff. Thus the proof by Bourget, that $J^n(x) = 0$ and $J^{n+p}(x) = 0$, if n and p are integers, have no common root, is declared to be valid only for the case of multiple roots, a proof also criticised by Rayleigh.

A cylindrical harmonic is defined as a solution of two functional equations which are shown to lead to Bessel's equation. The general solution of the first fundamental equation is obtained, and from a new property of the second equation follows a solution in the form of a continued fraction. The influence which Kepler's equation has exerted upon the study of cylindrical harmonics is recognized, but the problem from which the functions obtained their name appears to have been overlooked. The author would advise dropping the name Bessel function, because Bessel used only integral parameters, also because Bernoulli, Euler, Laplace, Fourier, Poisson and others had previously known of them; but he has consented to call J^n the Bessel cylindrical harmonic and Y^n the Neumann cylindrical harmonic because of the fundamental work done by their respective investigators.

Dr. Nielsen has aimed to obtain generalized forms and theorems, and with this in view he has devoted considerable space in the first of the four parts of his work to Lommel's Π function and to the similar Φ function, thus laying a firm foundation for a new theory of definite integrals with cylindrical harmonics and for Schlömilch's series. In this part are given also new developments of Krampe's integral $\int_0^x \epsilon^{-x^2}dx$, and two of $J^n(x)$.

A chapter is also given to the integration of Riccati's equation, an equation of the third order and one of the fourth order by methods which may be extended to those of higher order.

Although this treatise does not go into practical applications, the author deviates from this course by giving at the end of the first part a list and brief description of tables of Bessel's functions mostly J^0 and J^1, including one of J^n, also one of Y^0 and Y^1. He might have included a table by Dr. Meissel giving the first fifty roots of $J'(x) = 0$ and the corresponding maximum or minimum values of $J^0(x)$, which may be found in Gray and Mathews's Bessel Functions, together with the formulas by Professor J. McMahon in this connection.

The new theory of definite integrals mentioned above is given in the second part, and though it would have been possible to express the results as particular integrals of the general differential equation, the author has preferred not to do so, and has thus avoided introducing a large class of more general functions. Attention is called to an incorrect equation by Struve which is quoted by Gray and Mathews, page 238, example 46. The latter have made one correction, but the denominator of the integrand as printed should be raised to the nth power.

The third part is devoted to the development of analytic functions by means of cylindrical harmonics and gives a new analogy between Neumann's series of the first and second kind; also two new developments of elliptic integrals of the first kind. An interesting result is a new addition formula, $J^n(x + y)$ in an infinite series of terms each containing a product of two Bessel functions.

The development of arbitrary functions in terms of cylindrical harmonics is the topic of the fourth part. A series expansion by Lommel, proceeding according to Bessel functions of ascending orders, each differentiated the same number of times, does not appear. A new solution of Kepler's equation is given, believed to have advantages in practical application.

At the end of the book is a collection of more than sixty formulas or theorems under the headings: gamma functions, hypergeometric series, spherical harmonics, etc., also a few notes upon portions of the text. These theorems are referred to by number in order to avoid digressions in the course of the proofs. Here appears a note concerning an important misprint in Dini's discussion of Fourier's series, referring also to a letter from Dini. Earlier in the text Dini was mentioned as the only one among several writers who had given a rigorous proof of the correctness of the expansion of $f(x)$ in a series of terms each involving a Bessel function. This expansion caused Todhunter to state that many German writers credit Fourier with its authorship, though in fact he did not give it; and into this category it may be inferred that Nielsen has fallen.

Last of all comes a most important part of the book, a very complete bibliography, giving references to both theoretical and applied work in cylindrical harmonics. In addition, at the bottom of many pages are references to the original sources of nearly all formulas, in many cases proved by methods different from those in the text. A paper by Glaisher on Riccati's equation appeared in *Philosophical Transactions* in 1881, not in 1882, while to Schläfli's credit may be added an extensive article in *Annali di Matematica*, series 2, volume 6. In 1867 Lommel mentioned nine writers, while in this list appear one hundred and fifty-five.

With all the work which Dr. Nielsen has brought within the compass of a volume of moderate size, and in which he has had so great a share, there remain unexplored fields. Apart from his frank statement that we do not know the necessary and sufficient conditions under which a function is developable in a Fourier's series, there are other topics more closely related to the text, such as the remainder terms in null developments, and the single valuedness of developments in a Schlömilch's series, also many topics not fully treated in the present work.

F. H. SAFFORD.

Space and Geometry in the Light of Physiological, Psychological and Physical Inquiry. By Dr. ERNST MACH. Translated from the German by THOMAS J. MCCORMACK. Chicago, The Open Court Publishing Company, 1906. 148 pp.

To appreciate this work it is necessary to view it in its relation to two complementary movements in modern mathematical thought, namely, the logical movement and another that may be significantly called biological. The aim of the former has been to detect and to enumerate all definite ideas or terms that are indefinable and all definite propositions that are indemonstrable,