NOTE ON THE NUMERICAL TRANSCENDENTS S_n AND $s_n = S_n - 1$.

BY PROFESSOR W. WOOLSEY JOHNSON.

1. The numbers defined by the series

$$S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots,$$

where n is a positive integer greater than unity, are of frequent occurrence in analysis. Euler, in the Institutiones Calculi Differentialis, 1775, gave a table of their values up to S_{16} to sixteen places of decimals (page 456). He had connected the values of the even-numbered ones with Bernoulli's numbers and the even powers of π by the formula

$$S_{2n} = \frac{2^{2n-1} B_n \pi^{2n}}{(2n)!},$$

but failed to obtain a finite expression for the odd-numbered ones. He gave also the value of the constant γ, now known as Euler's constant, defined as the limiting value

$$\gamma = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n \right]_{n=\infty}.$$

2. The constants γ and S_n occur in the expressions for the function $\log \Gamma(1+x)$; and Legendre, for the purpose of constructing his table of $\log \Gamma(a)$, examined Euler's table of values for S_n and, finding "quelques erreurs assez graves," reconstructed the table, carrying it to S_{35}, remarking that for higher values of n one has only to divide the excess over unity successively by 2. This is of course because the powers of 1/3, 1/4, etc., have disappeared from the last retained decimal place, which was as in Euler's table the sixteenth.

The values of S_n rapidly approach that of the first term, which is unity. It follows that, an algebraic series being given in which the S_n's occur, if the series resulting from replacing S_n by unity has been already summed, we can by subtraction obtain a much more rapidly convergent series. Thus Legendre replaced his series

$$\log \Gamma(1+x) = \tfrac{1}{2}\log\frac{\pi x}{\sin \pi x} - \gamma x - \tfrac{1}{3}S_3 x^3 - \tfrac{1}{5}S_5 x^5 - \cdots$$

by

$$\log \Gamma(1+x) = \tfrac{1}{2}\log\frac{\pi x}{\sin \pi x} - \tfrac{1}{2}\log\frac{1+x}{1-x}$$
$$+ (1-\gamma)x - \tfrac{1}{3}(S_3 - 1)x^3 - \tfrac{1}{5}(S_5 - 1)x^5 - \cdots.$$

3. I propose in this note to put $S_n = 1 + s_n$, so that

$$s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots.$$

A table of values of S_n is of course virtually a table of values of s_n.

When values of x for which the Γ-function is known are substituted in the algebraic series, numerical series involving S_n and γ are found; thus Legendre * derives from the above equation by putting $x = 1$ and $x = \frac{1}{2}$,

(1) $$1 - \gamma = \tfrac{1}{2}\log 2 + \tfrac{1}{3}s_3 + \tfrac{1}{5}s_5 + \tfrac{1}{7}s_7 + \cdots,$$

(2) $$1 - \gamma = \log \tfrac{3}{2} + \frac{s_3}{3 \cdot 2^2} + \frac{s_5}{5 \cdot 2^4} + \frac{s_7}{7 \cdot 2^6} + \cdots.$$

Legendre computed γ from each of these series by means of his table of S_n, and cites the agreement with the known value of γ as a test of the accuracy of the table.

Dr. Glaisher in a paper "On the history of Euler's constant," *Messenger of Mathematics*, 1871, cites a number of these relations from the Memoirs of Euler, one of which is, in the present notation,

(3) $$\gamma = \tfrac{1}{2}s_2 + \tfrac{2}{3}s_3 + \tfrac{3}{4}s_4 + \cdots.$$

This formula was given in 1769, and in a memoir of 1781 occurs the formula

(4) $$1 - \gamma = \tfrac{1}{2}s_2 + \tfrac{1}{3}s_3 + \tfrac{1}{4}s_4 + \cdots.$$

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* Traité des fonctions elliptiques et des intégrales eulériennes, vol. 2, p. 434. The table of values of S_n is on p. 432.

4. It was a comparison of these equations which first led me to notice the very simple relation, independent of γ,

$$(5) \qquad 1 = s_2 + s_3 + s_4 + \cdots.$$

This relation must doubtless have been frequently noticed by those who have had occasion to deal with these numbers,* yet it seems clear that it was not known to Legendre writing in 1826, for it forms a much better verification of his table (involving, as it does, every figure of it) than does the computation of γ mentioned above.

5. It is the main object of this note to point out that not only this result but the other results mentioned above involving γ and naperian logarithms may be derived by direct summation of series in s_n's.

Thus, if we write out the terms of each s_n in a column, we have

$$\begin{aligned}\sum_2^\infty s_n = & \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 + \cdots \\ & + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \left(\tfrac{1}{3}\right)^4 + \cdots \\ & + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^3 + \cdots \\ & + \cdots.\end{aligned}$$

The rows form geometric progressions, hence we have

$$\begin{aligned}\sum_2^\infty s_n &= \frac{\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} + \frac{\left(\frac{1}{3}\right)^2}{1-\frac{1}{3}} + \frac{\left(\frac{1}{4}\right)^2}{1-\frac{1}{4}} + \cdots \\ &= \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \cdots = 1.\end{aligned}$$

In like manner we can show that

$$s_2 - s_3 + s_4 - \cdots = \tfrac{1}{2};$$

whence it follows that

$$(6) \qquad s_2 + s_4 + s_6 + \cdots = \tfrac{3}{4},$$

* I have, since writing the above, found it given as a problem in the second edition of Boole's Finite Differences in the form: "Shew that the sum of all the negative powers of all whole numbers (unity being in both cases excluded) is unity; if odd powers are excluded, it is $\tfrac{3}{4}$."

and

(7) $$s_3 + s_5 + s_7 + \cdots = \tfrac{1}{4}.$$

Legendre's table is completely verified by these equations, the sums being in fact only 2 or 3 units out in the 16th place.

Again, to sum $\frac{1}{2}s_2 + \frac{1}{3}s_3 + \cdots$, we have

$$\begin{aligned}\sum_2^\infty \frac{s_n}{n} &= \tfrac{1}{2}(\tfrac{1}{2})^2 + \tfrac{1}{3}(\tfrac{1}{2})^3 + \tfrac{1}{4}(\tfrac{1}{2})^4 + \cdots \\ &\quad + \tfrac{1}{2}(\tfrac{1}{3})^2 + \tfrac{1}{3}(\tfrac{1}{3})^3 + \cdots \\ &\quad + \tfrac{1}{2}(\tfrac{1}{4})^2 + \tfrac{1}{3}(\tfrac{1}{4})^3 + \cdots \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

Here each row is of the form

$$\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \log\frac{1}{1-x} - x,$$

where x has the values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., in the successive rows. Hence

$$\begin{aligned}\sum_2^\infty \frac{s_n}{n} &= \Big[\log 2 + \log\tfrac{3}{2} + \log\tfrac{4}{3} + \cdots + \log\frac{m}{m-1} \\ &\qquad\qquad - \left(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\right)\Big]_{m=\infty} \\ &= \left[\log m - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\right)\right]_{m=\infty} + 1 = 1 - \gamma,\end{aligned}$$

which is equation (4) above.

6. Treating the sum $s_2 + \frac{1}{2}s_4 + \frac{1}{3}s_6 + \cdots$ in the same manner, the rows are of the form

$$x^2 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^6 + \cdots = \log\frac{1}{1-x^2},$$

x having the successive values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. Thus

(8) $$\begin{aligned}\sum_1^\infty \frac{s_{2n}}{n} &= \log\tfrac{4}{3} + \log\tfrac{9}{8} + \log\tfrac{16}{15} + \cdots \\ &= \log\left[\tfrac{2}{1}\ \tfrac{2}{3}\ \tfrac{3}{2}\ \tfrac{3}{4}\ \tfrac{4}{3}\ \tfrac{4}{5} \cdots\right] = \log 2.\end{aligned}$$

As a further example, Legendre's equation, (2) of this note, may be verified. Thus in summing $\frac{s_3}{3.2^2} + \frac{s_5}{5.2^4} + \cdots$, the rows take the form, if $y = \frac{1}{2}x$,

$$\frac{x^3}{3\cdot 2^2} + \frac{x^5}{5\cdot 2^4} + \frac{x^7}{7\cdot 2^6} = 2\left[\frac{y^3}{3} + \frac{y^5}{5} + \cdots\right] = \log\frac{1+y}{1-y} - 2y$$

where y takes the successive values $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, etc. Hence

$$\begin{aligned}\sum_1^\infty \frac{s_{2n+1}}{(2n+1)2^{2n}} &= \log\left[\tfrac{5}{3}\ \tfrac{7}{5}\ \tfrac{9}{7}\cdots\tfrac{2n+1}{2n-1}\right]_{n=\infty} \\ &\qquad - \left[\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n}\right]_{n=\infty} \\ &= \Big[\log(2n+1) - \log 3 + 1 \\ &\qquad - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}\right)\Big]_{n=\infty}\end{aligned}$$

and, since $\log(2n+1)$ at the limit becomes $\log 2 + \log n$, the result is

$$\sum_1^\infty \frac{s_{2n+1}}{(2n+1)2^{2n}} = \log\tfrac{2}{3} + 1 - \gamma,$$

which is equation (2).

7. I have been tempted to make an independent calculation of the values of s_n, using 12 decimal places. The values of s_n when $n > 9$, when carried thus far, are readily obtained by direct summation of all their terms. But the extreme slowness of the convergence of the terms when n is small renders it practically necessary to employ the same method as that by which γ has been computed, namely the Euler-Maclaurin formula for the summation of a finite number of terms, viz.:

$$\Sigma u_x = C + \int u_x dx - \tfrac{1}{2}u_x + \frac{B_1}{2}\frac{du_x}{dx} - \frac{B_2}{4!}\frac{d^3u_x}{dx^3} + \frac{B_3}{6!}\frac{d^5u_x}{du^5} - \cdots$$

In the application of this formula when u_x is a negative power of x, the constant C is the sum to infinity, and is obtained by direct summation of Σu_x to a moderate value of x, and calculation of the infinite series in the second member. It was thus that Euler and Legendre calculated the value of γ as well as the values of S_n, and thus that Adams, after greatly extending the range of known Bernoullian numbers, calculated γ to 263 place of decimals.

Inasmuch as Legendre's table has not often been reprinted, it may be of interest to give the results of my computation to the eleventh place of decimals. They are as follows:

$$\text{Values of } s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$$

n	s_n	n	s_n
2	.64493 40668 5	14	.00006 12481 4
3	.20205 69031 6	15	3 05882 4
4	8232 32337 1	16	1 52822 6
5	3692 77551 4	17	76372 0
6	1734 30619 8	18	38172 9
7	834 92773 8	19	19082 1
8	407 73562 0	20	9539 6
9	200 83928 3	21	4769 3
10	99 45751 3	22	2384 5
11	49 41886 0	23	1192 2
12	24 60865 5	24	596 1
13	12 27133 5	25	298 0

[The values for $n > 25$ are obtained each by dividing its predecessor by 2.]

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ON CERTAIN PROPERTIES OF WRONSKIANS AND RELATED MATRICES.

BY PROFESSOR D. R. CURTISS.

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In this note I shall present theorems of a very general character on the vanishing of Wronskians and related matrices. Proofs, however, will be reserved for subsequent publication in more extended form.

Let $u_1, u_2, \cdots, u_n$ be functions, real or complex, of the real variable x, having finite derivatives of the first k orders