10. If l cuts c_4 in κ, the surface is a nodal cubic, having one node at κ, and another at the node of c_4. R_8 breaks up into K_3 and R_5, the latter having the symbol $l_2 + c_4^2 + 3^2$ (Schwarz's A, vii), the double generation being the line joining the node to the fourth point in the plane containing l. Thus,

$$(F_3, R_5) = l(2) + c_4(8) + g_2(2) + 3s_2(3).$$

If c_4 has a cusp, the second nodal point becomes uniplanar. Further specializations result in quadrics and quadric cones.

CORNELL UNIVERSITY,
January, 1906.

OPERATION GROUPS OF ORDER $p_1^{m_1\mu_1}p_2^{m_2\mu_2}$.

BY PROFESSOR O. E. GLENN.

IT is desired to make certain generalizations concerning the groups of order the product of powers of two primes p_1, p_2, such that $p_1 \equiv 1 (\bmod p_2)$, these groups possessing abelian subgroups H_i of type $[\mu_i, \mu_i, \cdots, \mu_i]$ $(i = 1, 2)$. It is possible to specify for these groups those subgroups (here called basic subgroups) from which it is necessary and sufficient that generating operations be selected in order that they may generate the whole group G. This general problem connected with groups of composite order seems to merit more attention than it has thus far received. If

$$H_1 = \{P_1, P_2, \cdots, P_{m_1}\}, \quad H_2 = \{Q_1, Q_2, \cdots, Q_{m_2}\},$$

then the number of operations of order $p_i^{\mu_i}$ in H_i is

$$\sum_{j=0}^{m_i-1} {}_{m_i}C_j [p_i^{\mu_i} - \Phi(p_i^{\mu_i})]^j [\Phi(p_i^{\mu_i})]^{m_i-j}$$
$$= [p_i^{\mu_i-1} + \Phi(p_i^{\mu_i})]^{m_i} - p^{m_i(\mu_i-1)},$$

so that the number of cyclical subgroups of order $p_i^{\mu_i}$ in H_i is

$$N_{p_i}^{\mu_i} = \frac{p_i^{m_i(\mu_i-1)}(p_i^{m_i} - 1)}{\Phi(p_i^{\mu})}$$
$$= p_i^{(m_i-1)(\mu_i-1)}(p_i^{m_i-1} + p_i^{m_i-2} + \cdots + p_i + 1).$$

Let us now suppose that $\mu_1 = \mu_2 = 1$. Transformation of the $N^1_{p_1}$ subgroups of H_1 by Q_κ permutes them in sets of p_2 or else leaves them invariant. If $\rho_{Q\kappa, p_1}$ is the number left invariant

$$N^1_{p_1} - \rho_{Q\kappa, p_1} \equiv 0 \pmod{p_2},$$

and when $p_1 \equiv 1 \pmod{p_2}$, $\rho_{Q_\kappa, p_1} \geqq m_1$. It follows that P_λ ($\lambda = 1, 2, \cdots, m_1$) can be so selected that

$$(1) \qquad Q_1^{-1} P_\lambda Q_1 = P_\lambda^{\alpha^{x_{1\lambda}}}, \quad \alpha^{p_2} \equiv 1 \pmod{p_1}.$$

Now H_1 is self-conjugate * in G hence

$$Q_2^{-1} P_\lambda Q_2 = P_1^{a_{1\lambda}} P_2^{a_{2\lambda}} \cdots P_{m_1}^{a_{m_1\lambda}},$$

and

$$(Q_1 Q_2)^{-1} P_\lambda (Q_1 Q_2) = \prod_{\nu=1}^{m_1} P_\nu^{a_{\nu\lambda}\alpha^{x_{1\lambda}}} = (Q_2 Q_1)^{-1} P_\lambda (Q_2 Q_1)$$
$$= \prod_{\nu=1}^{m_1} P_\nu^{a_{\nu\lambda}\alpha^{x_{1\nu}}}.$$

$$(2) \qquad a_{\nu\lambda}(\alpha^{x_{1\lambda}} - \alpha^{x_{1\nu}}) \equiv 0 \pmod{p_1},$$

and if $x_{1\lambda} \not\equiv x_{1\nu} \pmod{p_2}$ then $a_{\nu\lambda} \equiv 0 \pmod{p_1}$. Suppose on the other hand that some of the $x_{1\lambda}$ are congruent to some of the $x_{1\nu}(\nu \not\equiv \lambda)$, say $x_{1\lambda} \equiv x_{1\nu} (\nu = \lambda + 1, \lambda + 2, \cdots, \lambda + r)$, then $a_{\lambda+i}$, may be zero or not. Assuming the latter, we prove that *all the subgroups in the group* $\{P_\lambda, P_{\lambda+1}, P_{\lambda+2}, \cdots, P_{\lambda+r}\}$ *are permutable with* Q_1, *and the operations* $P_\lambda, P_{\lambda+1}, P_{\lambda+2}, \cdots, P_{\lambda+r}$, Q_2 *form a subgroup of order* $p_1^{r+1} p_2$.

The first statement is obvious since

$$Q_1^{-1} \prod_{i=0}^{r} P_{\lambda+i}^{a_i} Q_1 = \left(\prod_{i=0}^{r} P_{\lambda+i}^{a_i} \right)^{\alpha^{x_{1\lambda}}}.$$

To prove the latter we have

$$Q_2^{-1} P_\lambda Q_2 = P_\lambda^{a_{\lambda\lambda}} P_{\lambda+1}^{a_{\lambda+1,\lambda}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda}} = \prod_{i=0}^{r} P_{\lambda+i}^{a_{\lambda+i,\lambda}}.$$

Also

$$(Q_1 Q_2)^{-1} P_{\lambda+i} (Q_1 Q_2) = P_1^{a_{1,\lambda+i}\alpha^{x_{1\lambda}}} P_2^{a_{2,\lambda+i}\alpha^{x_{1\lambda}}} \cdots P_{m_1}^{a_{m_1\lambda+i}\alpha^{x_{1\lambda}}}$$
$$= \prod_{j=1}^{m_1} P_j^{a_{j,\lambda+i}\alpha^{x_{1\lambda}}} = (Q_2 Q_1)^{-1} P_{\lambda+i} (Q_2 Q_1)$$
$$= P_1^{a_{1,\lambda+i}\alpha^{x_{11}}} P_2^{a_{2,\lambda+i}\alpha^{x_{12}}} \cdots P_\lambda^{a_{\lambda,\lambda+i}\alpha^{x_{1\lambda}}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda+i}\alpha^{x_{1\lambda}}} \cdots P_{m_1}^{a_{m_1\lambda+i}\alpha^{x_{1m_1}}},$$

* Burnside, Finite Groups, p. 351, Theorem V.

and since $$x_{1k} \not\equiv x_{1\lambda} \quad (k \neq \lambda, \lambda+1, \cdots, \lambda+r),$$

$$a_{j,\lambda+i} \equiv 0 \ (\mathrm{mod}\ p_1) \quad (j \neq \lambda, \lambda+1, \cdots, \lambda+r),$$

so that

$$Q_2^{-1} P_{\lambda+i} Q_2 = P_\lambda^{a_{\lambda,\lambda+i}} P_{\lambda+1}^{a_{\lambda+1,\lambda+i}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda+i}} = \prod_{j=\lambda}^{\lambda+r} P_j^{a_{j,\lambda+i}}$$
$$(i = 0, 1, 2, \cdots, r),$$

and the operations P_j, Q_2 form a subgroup (I) of order $p_1^{r+1}p_2$. The number of subgroups of order p_1 in (I) is

$$p_1^r + p_1^{r-1} + \cdots + p_1 + 1,$$

and at least $r+1$ of them are permutable with Q_2, since $p_1 \equiv 1 \ (\mathrm{mod}\ p_2)$. If these are $\{P'_j\}$ $(j = \lambda, \lambda+1, \cdots, \lambda+r)$, then, on replacing P_j by P'_j, the series of m_1 independent generators of H_1

$$P_1, P_2, P_3, \cdots, P_{\lambda-1}, P'_\lambda, P'_{\lambda+1}, \cdots, P'_{\lambda+r}, P_{\lambda+r+1}, \cdots, P_{m_1}$$

has the property that each generator is transformed into one of its own powers by both Q_1 and Q_2. Hence, referring back to equation (2), even when $x_{1\lambda} \equiv x_{1\nu}$ (mod p_2) we may write $a_{\nu\lambda} \equiv 0$ (mod p_1) $(\nu \neq \lambda)$. It is also obvious that when $\nu = \lambda$, then $a_{\lambda\lambda} \equiv$ some power of α, as $\alpha^{x_{2\lambda}}$. Hence G is defined by the relations

$$P_\lambda^{p_1} = Q_\kappa^{p_2} = 1,\ P_\lambda P_\nu = P_\nu P_\lambda,\ Q_\kappa Q_l = Q_l Q_\kappa,\ Q_\kappa^{-1} P_\lambda Q_\kappa = P_\lambda^{\alpha^{x_{\kappa\lambda}}}$$
$$(\lambda, \nu = 1, 2, \cdots, m_1;\ \kappa, l = 1, 2, \cdots, m_2)$$
$$\alpha^{p_2} \equiv 1 (\mathrm{mod}\ p_1), \quad x_{\kappa\lambda} \equiv 0, 1, \cdots, p_2 - 1 \ (\mathrm{mod}\ p_2).$$

LEMMA: *There are $m_2 - 1$ of the $x_{\kappa\lambda}$ equal to zero, and no generality is lost by assuming that*

$$x_{11} \equiv x_{21} \equiv x_{31} \equiv \cdots x_{m_2-1,1} \equiv 0,\ x_{m_2 1} \equiv 1 \ (\mathrm{mod}\ p_2).$$

Since G is by assumption non-divisible, there is at least one operation, as Q_{m_2}, which transforms P_1 into $P_1^\alpha (\alpha \neq 1)$. Say $Q_{m_2}^{-1} P_1 Q_{m_2} = P_1^\alpha$. If $Q'_{m_2-\tau}$ is any other generator of H_2, by what has been given

$$Q'^{-1}_{m_2-\tau} P_1 Q'_{m_2-\tau} = P_1^{\alpha^{x'_{m_2-\tau,1}}} = Q_{m_2}^{-x'_{m_2-\tau,1}} P_1 Q_{m_2}^{x'_{m_2-\tau,1}},$$

$$\therefore \left(Q'_{m_2-\tau} Q_{m_2}^{-x'_{m_2-\tau,1}} \right)^{-1} P_1 \left(Q'_{m_2-\tau} Q_{m_2}^{-x'_{m_2-\tau,1}} \right) = P_1,$$

$$(= Q_{m_2-\tau}^{-1} P_1 Q_{m_2-\tau} \text{ say}).$$

It follows that $Q_{m_2-\tau}$ $(\tau = 1, 2, \cdots, m_2 - 1)$ is permutable with P_1, so that $x_{11} \equiv x_{21} \cdots \equiv x_{m_2-1,1} \equiv 0$ and $x_{m_21} \equiv 1$. By this means all the generators of order p_2 of the group G are fixed and no other invariants $x_{\kappa\lambda}$ can be placed equal to zero without destroying the generality of the defining relations.

The most general form of the generating operations of order p_1 of G will now be derived.

Let

$$\bar{P}_t = P_1^{\alpha_{1t}} P_2^{\alpha_{2t}} \cdots P_{m_1}^{\alpha_{m_1t}}; \quad \bar{Q}_u = Q_1^{\beta_{1u}} Q_2^{\beta_{2u}} \cdots Q_{m_2}^{\beta_{m_2u}},$$

and write

$$\bar{Q}_u^{-1} \bar{P}_t \bar{Q}_u = \bar{P}_t^{\alpha^{x_{ut}}}.$$

Under the assumption $\alpha_{it} \not\equiv 0 \pmod{p_1}$ we get

$$(3) \quad \beta_{m_2u} x_{m_2i} + \beta_{m_2-1u} x_{m_2-1i} + \beta_{m_2-2u} x_{m_2-2i} + \cdots + \beta_{1u} x_{1i} \equiv x_{ut} \pmod{p_2} \quad (i = 1, 2, \cdots, m_1),$$

so that for a chosen t (say $t = 1$) and a chosen u (as $u = 1$) we have a set of m_1 linear congruences in the β_{ku} (as the following):

$$(4) \qquad \left.\begin{array}{l} \beta_{m_21} \equiv 0, \\ \beta_{m_2-11} x_{m_2-1,2} + \cdots + \beta_{11} x_{12} \equiv 0, \\ \beta_{m_2-11} x_{m_2-1,3} + \cdots + \beta_{11} x_{13} \equiv 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \beta_{m_2-11} x_{m_2-1,m_1} + \cdots + \beta_{11} x_{1m_1} \equiv 0, \end{array}\right\} \pmod{p_2}.$$

If no $\alpha_{i1} \equiv 0$, then assuming $m_1 = m_2$,

$$\begin{vmatrix} x_{m_2-1,2} & \cdots & x_{12} \\ x_{m_2-1,3} & \cdots & x_{13} \\ \cdot & \cdot \ \cdot \ \cdot & \cdot \\ x_{m_2-1,m_1} & \cdots & x_{1m_1} \end{vmatrix} \equiv 0 \pmod{p_2},$$

a relation among the invariants of the group which is not possible *in general.* Hence since we seek *general* generators of the group we must have at least one α_{i1} ($\not\equiv a_{11}$ since $\bar{P}_1$ obviously $= P_1^{\alpha_{11}}$ as a special case) and one of the β_{k1}; viz., β_{m_21} equal to zero. Again if $m_1 = m_2 + \sigma$ ($\sigma =$ integer), then at least $\sigma + 1$ of the α_{i1} ($\not\equiv \alpha_{11}$) are zero. If $m_1 < m_2$, no α_{i1} is necessarily zero, by virtue of (4). In the same way if $u = 2$, $t = 1$, $\beta_{m_2, 2} \equiv 0$, and in general if $u = u(< m_2)t = 1$, $\beta_{m_2u} \equiv 0$; the same condition holding for $m_1 \gtreqless m_2$ as above. When $u = m_2$, $t = 1$ we have

$$\beta_{m_2m_2}x_{m_21} \equiv 1, \text{ (whence } \beta_{m_2m_2} \equiv 1 \pmod{p_2}).$$

$$\left.\begin{array}{l} x_{m_22} + \beta_{m_2-1m_2}x_{m_2-12} + \cdots + \beta_{1m_2}x_{12} \equiv 1, \\ x_{m_23} + \beta_{m_2-1m_2}x_{m_2-13} + \cdots + \beta_{1m_2}x_{13} \equiv 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{m_2m_1} + \beta_{m_2-1m_2}x_{m_2-1m_1} + \cdots + \beta_{1m_2}x_{1m_1} \equiv 1, \end{array}\right\} \pmod{p_2}.$$

And except for $\beta_{m_2m_2} \equiv 1$, the conclusions are the same as above; and the argument may be repeated for $t = 2, 3, \ldots, m_1$ in equation (3). The general result is the following:

THEOREM: *The necessary and sufficient conditions that a given set of m_i operations of order p_i $(i = 1, 2)$ of G shall generate G are* (1) *that they be independent,* (2) *if $m_1 = m_2 + \sigma$ then $\overline{P}_t$ must be selected from a certain basic subgroup of H_1, of order $\leqq p_1^{m_1-\sigma-1}$, in which of course P_t occurs; $\overline{Q}_u(u<m_2)$ is likewise an operation of a subgroup of order $p_2^{m_2-1}$, containing Q_u, viz., of $\{Q_1, Q_2, \cdots, Q_{m_2-1}\}$. It immediately follows since the $x_{\kappa\lambda}$ are invariants of the group that these basic subgroups must be of order p_1 (one $a_{it} \not\equiv 0$) and coincide with the $\{P_t\}$. The number of groups isomorphic to a given type (given set of invariants $x_{k\lambda}$) is equal to the number of distinct types obtained by transforming G by all $m_1!$ permutations of the basic subgroups.*

In particular if $m_2 = 1$, so that $m_1 = m_2 + m_1 - 1$ the basic subgroups are of order $p_1^{m_1-m_1+1} = p_1$. In other words $\overline{P}_t = P_t$ and $\overline{Q}_1 = Q_1^{\beta_{11}}$, and all the types isomorphic to a given type are obtainable by permuting the m_1 subgroups $\{P_t\}$ in the $m_1!$ possible ways. The number of non-isomorphic types has been determined for this case in the form*

*BULLETIN, vol. 11, No. 6, p. 318.

$$N=\frac{1}{m_1}\left[\sum_{\sigma=0}^{(m_1-1)(p_2-2)} P(0,1,\cdots,p_2-2)^{m_1-1}\sigma+\psi\right],$$

where $P(0, 1, \cdots, p_2 - 2)^{m_1-1}\sigma$ stands for the number of partitions of σ in $(m_1 - 1)$'s by the numbers $0, 1, \cdots, p_2 - 2$; and ψ is a determinate function of p_2 and m_1.

SPRINGFIELD, MO.,
December, 1905.

A DEFINITION OF QUATERNIONS BY INDEPENDENT POSTULATES.*

BY MISS R. L. CARSTENS.

(Read before the American Mathematical Society, February 24, 1906.)

§ 1. *Quaternions with respect to a Domain D.*†

THE usual theory relates to quaternions $a_1 + a_2i + a_3j + a_4k$ in which the coefficients a_i range independently over all real numbers or else over all complex numbers, and the units have the following multiplication table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

These conditions give the real quaternion system and the octonion system.‡ As an obvious generalization, the coefficients may range independently over all the elements of any domain D.

---

*See Dickson, "On hypercomplex number systems," *Transactions Amer. Math. Society*, vol. 6 (1905).

† A domain consists of any class of elements.

‡ Octonions may be considered as quaternions with complex coefficients.