RATIONAL PLANE CURVES RELATED TO RIEMANN TRANSFORMATIONS.

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IN a plane considered as filled with algebraic curves of all orders and deficiencies, a perfectly arbitrary net or doubly infinite linear system of such curves may be regarded as effecting a one-to-one transformation. For exceptional curves the transformation may be two-to-one (or m-to-one), and such curves are termed in another connection involution curves of the net. Only in case the net consists of rational curves, having but a single variable point of intersection is the transformation a Cremona transformation. In other cases, barring all involution curves of the net, it is called a Riemann transformation, or a birational transformation of the curve under consideration. Let the x-plane be transformed into the y-plane by the substitution

$$\rho y_1 = \phi_1(x_1, x_2, x_3) = \phi_1(x), \quad \rho y_2 = \phi_2(x), \quad \rho y_3 = \phi_3(x);$$

and let $f(x) = 0$ represent a curve which is transformed into $F(y) = 0$. To every point (y) on the curve F (with a finite number of exceptions) let only one point (x) correspond, then the transformation is birational for that curve; and we may call it a Riemann transformation of the plane, as proposed by Dr. Kasner. If the curves $\phi_1(x) = 0$, $\phi_2(x) = 0$, $\phi_3(x) = 0$ are rational of order n, with multiple points of like structure at fixed points, and enough other fixed simple base points to leave only one intersection variable, then we have a Cremona transformation of the entire plane. My purpose is to call attention to a peculiarity of rational curves under Riemann transformations, viz., to show that each possesses, for every Riemann transformation, one of the Cremona type which is on that curve equivalent to the former.

Begin with a straight line U, or $(ux) = 0$. Replace the Riemann substitution by the following:

$$\begin{aligned}\rho y_1 &= \phi_1(x) + \psi_1(x)\cdot(ux) \equiv \Phi_1(x),\\ \rho y_2 &= \phi_2(x) + \psi_2(x)\cdot(ux) \equiv \Phi_2(x),\\ \rho y_3 &= \phi_3(x) + \psi_3(x)\cdot(ux) \equiv \Phi_3(x),\end{aligned}$$

and take ψ_1, ψ_2, ψ_3 any forms of order $n-1$ to make the expressions on the right homogeneous. For every point x on $(ux)=0$ this gives the same point (y) as before, *i. e.*, it is equivalent to the former on the line U. The net of Φ-curves can now be made a Cremona net of Jonquières type. For each ψ contains $\frac{1}{2}n(n+1)$ constants linearly. To have a point of multiplicity $n-1$ in a specified point O is equivalent to $\frac{1}{2}(n-1)n$ linear conditions. Each ψ retains now n parameters: select those arbitrarily for one ψ, say ψ_1, then dispose of the remaining $2n$ so that the curves $\Phi_2=0$ and $\Phi_3=0$ may have on $\Phi_1=0$ $2n-2$ additional points of intersection in common, none of them on the line U. The transformation is now of the Cremona type, as was required.

On every straight line, a Riemann transformation is conformable with an indefinitely great number of Cremona transformations.

Next consider any rational curve of order above the first, $f(x)=0$. Find a Cremona transformation S which converts f into the line U, and let S^{-1} denote its inverse. Let R denote the proposed Riemann transformation, call $SR \equiv R'$, and by the above theorem R' must be conformable along the line U to a Cremona transformation K. Then along the curve f, we must have $S^{-1}SR$ or $S^{-1}R'$ conformable with $S^{-1}K$ or K'; and evidently this is of the Cremona type, since Cremona transformations constitute a group. This shows that:

On any rational plane curve, not an involution curve of the net, a Riemann transformation R with a net of φ-curves is equivalent to or conformable with a Cremona transformation whose net of Φ-curves is of the Jonquières type.

Remembering that a rational curve has no invariants (or moduli) under algebraic transformation, we have learned that also it can have none under Cremona transformations; or negatively stated, the distinctive properties of the Cremona group within the group of Riemann transformations must be sought in their effect on systems of curves or on single curves of deficiency higher than zero.

The extension of this remark to space of more than two dimensions is not apparent. For verification, nets of conics and cubics have been employed, and no obstacle found to the actualization of the steps referred to in the above counting of constants.

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