au, bv, cw, dx meet at a point S in ω, the section of these planes by ω gives chords of the conic concurrent in S. Thus we have the theorem: If points of a conic are projectively paired, they lie on concurrent chords, which is the foundation of the theory of involution on a conic.

For the sake of brevity, as little detail as possible has been given in this note, the design being simply to draw attention to this mode of proving fundamental properties of the conic.

BRYN MAWR COLLEGE,
June, 1905.

ARZELÀ'S CONDITION FOR THE CONTINUITY OF A FUNCTION DEFINED BY A SERIES OF CONTINUOUS FUNCTIONS.

BY PROFESSOR E. J. TOWNSEND.

(Read before the American Mathematical Society, September 7, 1905.)

§ 1. A FUNCTION defined by a series whose terms are continuous functions may or may not be itself continuous. It may in fact be discontinuous at a set of points everywhere dense within the interval of definition. It is important to establish a criterion by which the continuity of such a function may be determined. Conditions which are sufficient, although not necessary, are to be found in any extensive work on calculus. Arzelà was the first to formulate a set of conditions which are both necessary and sufficient.* In his first discussion of the subject, however, he was not sufficiently rigorous.† A later and more rigorous development was given, differing from the first in some particulars.‡ Still more recently he has revised his first set of proofs and maintains that they are now sufficiently rigorous to be valid.§ It is the purpose of this paper to present in substance the final results of Arzelà's investigations.

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*Intorno alla continuità della somma di infinite funzioni continue, Bologna, 1884.

†See Schoenflies, Punktmannigfaltigkeiten, p. 225, footnote. Also Arzelà, Sulla serie di funzioni di variabili reali, Bologna, 1902, p. 6.

‡Sulla serie di funzioni, part 1, Bologna, 1899, p. 10 et seq.

§See Sulla serie di funzioni di variabili reali, Bologna, 1902.

Let us then define $S_n(x)$ by the relation

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_\kappa(x) + \cdots + u_n(x), \quad a \leqq x \leqq \beta,$$

where $u_\kappa(x)$ and hence $S_n(x)$ is a continuous function of x. Moreover, let us define $f(x)$ by the equation

$$f(x) = \mathop{\mathrm{L}}_{n=\infty} S_n(x), \qquad a \leqq x \leqq \beta.$$

The interval of convergence of the series is then the interval of definition of the function $f(x)$. The problem is to find a set of conditions which are necessary and sufficient for the continuity of $f(x)$ in a given interval (a, β), equal to or less than the interval of convergence.

With Arzelà, let us consider this problem as a special case of the following more general problem. Suppose $f(x, y)$ to be a function of the two independent variables x and y, defined for all values of x within the closed interval (a, β) and for

$$y \equiv (y) = y_1, \; y_2, \; y_3, \; \cdots, \; y_s, \; \cdots,$$

$y_1, y_2, y_3, \cdots, y_s, \cdots$ being a set of values dense at $y = y_0$, but not including y_0. Let $f(x, y_s)$ be a continuous function of x for each value y_s included in the set (y). For $y = y_0$, let $f(x, y)$ be defined by the equation

$$f(x, y_0) = \mathop{\mathrm{L}}_{y_s = y_0} f(x, y_s).$$

The more general problem, which we shall now consider, is to find a necessary and sufficient set of conditions that the limiting function $f(x, y_0)$ shall be a continuous function of x in the interval $a \leqq x \leqq \beta$. If we put $y_s = 1/n$, $n = 1, 2, 3, \cdots$, this reduces at once to the case of the infinite series given above.*

§ 2. First of all, let us consider the necessary and sufficient condition that the limiting function $f(x, y_0)$ shall be continuous at a single point x_0 of the interval (α, β). This condition may be stated as follows :

*The results given in this paper can be obtained, and perhaps more easily, without the introduction of the more general problem indicated. It is sufficient for this purpose to consider $S_n(x)$ as a function of the two variables x, n and to make of use the principles of ordinary convergence. Arzelà's method has been retained, not because it has any virtue in itself, but because it gives an opportunity to interpret more clearly his results at a point where he has been criticized.

I. *In order that the function* $f(x, y_0)$, *defined as in* § 1 *for the interval* (α, β), *be continuous at any point* x_0 *in this interval, it is necessary and sufficient that to an arbitrarily small positive number* σ *and to every* y_t *sufficiently near* y_0 *there shall correspond a neighborhood of the point* x_0, *which, however, may vary in extent with* y_t, *such that for every value of* x *in this neighborhood*

$$|f(x, y_0) - f(x, y_t)| < \sigma.$$

We shall first show that the above condition is necessary. Let us suppose then that $f(x, y_0)$ is a continuous function of x at the point $x = x_0$. By virtue of the limit

$$f(x_0, y_0) = \mathop{\mathbf{L}}_{y_s = y_0} f(x_0, y_s),$$

there must exist a definite number y_{s_1} of the set (y) such that for every y_t of this set lying between y_{s_1} and y_0, we have

$$(1) \qquad |f(x_0, y_0) - f(x_0, y_t)| < \sigma/3,$$

where σ as elsewhere in this paper is an arbitrarily small posi-

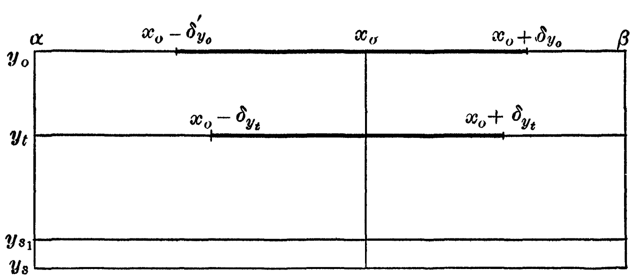

FIG. 1.

tive number. Because $f(x, y_0)$ is a continuous function of x for $y = y_0$, we have also for some neighborhood of x_0, say $(x_0 - \delta'_{y_0}, x_0 + \delta_{y_0})$, the inequality

$$(2) \qquad |f(x, y_0) - f(x_0, y_0)| < \sigma/3.$$

Likewise, because $f(x, y_t)$ is a continuous function of x, there exists a neighborhood $(x_0 - \delta'_{y_t}, x_0 + \delta_{y_t})$ on the line $y = y_t$ (Fig. 1), such that for every x within it

$$(3) \qquad |f(x_0, y_t) - f(x, y_t)| < \sigma/3.$$

Combining (1), (2), (3), we have

$$|f(x, y_0) - f(x, y_t)| < \sigma,$$

which holds for every y_t between y_{s_1} and y_0 and for every x within a neighborhood $(x_0 - d'_{y_t}, x_0 + d_{y_t})$, where d'_{y_t} is the smaller of the two values δ'_{y_0} and δ'_{y_t}, and d_{y_t} is the smaller of the two values δ_{y_0} and δ_{y_t}.

The given condition may be shown to be sufficient as follows. We have by hypothesis the inequality

$$(4) \qquad |f(x, y_t) - f(x, y_0)| < \sigma,$$

which is valid for every value of x within a certain neighborhood, say $(x_0 - \delta', x_0 + \delta)$. The existence of the limit

$$f(x_0, y_0) = \underset{y_t \doteq y_0}{\mathrm{L}} f(x_0, y_t)$$

gives, for all values of y_t from a certain definite value on, the inequality

$$(5) \qquad |f(x_0, y_0) - f(x_0, y_t)| < \sigma.$$

By hypothesis, $f(x, y_t)$ is a continuous function of x for each value of y_t. Hence, for every x within a certain neighborhood, say $(x_0 - \delta'_1, x_0 + \delta_1)$, we have

$$(6) \qquad |f(x_0, y_t) - f(x, y_t)| < \sigma.$$

By combining the inequalities (4), (5), and (6), we have

$$|f(x_0, y_0) - f(x, y_0)| < 3\,\sigma,$$

which holds for all values of x lying within the smaller of the two neighborhoods $(x_0 - \delta', x_0 + \delta)$ and $(x_0 - \delta'_1, x_0 + \delta_1)$. This inequality establishes the continuity of $f(x, y_0)$ at the point $x = x_0$ as the theorem requires.

§ 3. Let us now consider the necessary and sufficient condition that $f(x, y_0)$ shall be a continuous function of x throughout the closed interval (α, β). This condition Arzelà states in substance as follows:

II. *In order that the function* $f(x, y_0)$ *defined as in* § 1 *be continuous throughout the interval* $\alpha \leqq x \leqq \beta$, *it is necessary and* *sufficient that to an arbitrarily small positive number* σ *and to an arbitrary* $y_s \neq y_0$, *there shall correspond a positive number* l *and a finite set of values*

$$y = y_{s_1}, y_{s_2}, \cdots, y_{s_p}$$

lying between y_s *and* y_0, *for which the following condition holds: On lines* $y = y_{s_t} (t = 1, 2, \cdots, p)$, *it shall be possible to choose a series of segments, each of length at least equal to* l, *whose projections on the line* $y = y_0$, *when taken together, completely cover the interval* $\alpha \leqq x \leqq \beta$, *and for which, furthermore, the relation*

$$|f(x, y_0) - f(x, y_{s_t})| < \sigma$$

holds, (x, y_{s_t}) *being an arbitrary point on any one of these segments.**

It may be shown as follows that this condition is necessary.† Let us assume then that $f(x, y_0)$ is a continuous function of x throughout the interval $\alpha \leqq x \leqq \beta$. By virtue of theorem I, there exists for any $x'(\alpha \leqq x' \leqq \beta)$ and for each y_t sufficiently near y_0 a neighborhood $(x' - \delta'_{y_t}, x' + \delta_{y_t})$, which may vary with y_t but always, however, in such a manner that for each value of x within it the inequality

$$(1) \qquad |f(x, y_0) - f(x, y_t)| < \sigma$$

is valid for any previously assigned positive value of σ. In this discussion Arzelà distinguishes between that part of the neighborhood to the right of x' and that part lying to the left. Let us denote these two parts of the neighborhood by $\Delta(x', y_t)$ and $\Delta'(x', y_t)$ respectively. These two parts may or may not be of the same magnitude. In other words, having chosen an arbitrarily small positive number σ and a y_s at pleasure, we may then select any value x' of the interval (α, β), end points included, and there will exist upon some line $y = y_t$ lying between $y = y_s$ and $y = y_0$ a neighborhood of magnitude $\Delta(x', y_t)$ to the right of x', such that for every x within it the inequality (1) is valid. Let us now consider $\Delta(x', y_t)$ for all values of y_t included in the set (y) and lying between y_0 and y_s. For some of these values of y_t, $\Delta(x', y_t)$ may be zero, and consequently

* In his statement of the theorem, Arzelà does not say that the given interval shall be closed. This, however, is necessary.

† See Sulle serie di funzioni, Bologna, 1899, p. 17, et seq.

for these particular values of y_t no neighborhood exists to the right of x' for which the above inequality holds. It may happen that for some values of y_t lying between y_0 and y_s this inequality does not hold even for x' itself. However $\Delta(x', y_t)$ cannot be zero for all values of y_t lying between the above limits unless x' is the extreme point β; for we know that when y_t is taken sufficiently near y_0 $\Delta(x', y_t)$ is always greater than zero. Having selected x', there may exist for any particular y_t a great variety of values which might be taken as the corresponding $\Delta(x', y_t)$. In what follows, let us understand by $\Delta(x', y_t)$ the upper limit of all these values. This magnitude $\Delta(x', y_t)$, regarded as a function of y_t for all values of y_t between y_s and y_0, has an upper limit greater than zero, providing x' is different from β. Such a superior limit is therefore definitely determined for each value of x', σ and y_s having been previously selected. Let us denote this upper limit by $\Delta(x')$. In the same way, consider that part of the interval $(x' - \delta'_{y_t}, x' + \delta_{y_t})$ lying to the left of x', and denote by $\Delta'(x')$ the corresponding upper limit, which must likewise be different from zero for x' different from α.

Consider now the sum

$$\Delta'(x') + \Delta(x').$$

This sum is uniquely determined for each value x' of x, where $\alpha \leqq x \leqq \beta$. As a function of x, it has a lower limit which we shall now show to be greater than zero. Denote this lower limit by l. There must exist then in (α, β) at least one point, in every neighborhood of which the lower limit of $\Delta'(x) + \Delta(x)$ is l. Let x_1 be such a point. For the point x_1 itself, we have the value $\Delta'(x_1) + \Delta(x_1)$, which is certainly greater than zero. Among the values of (y) included between y_s and y_0, there is at least one, say y_t, for which the magnitude $\Delta(x_1, y_t)$ of the neighborhood to the right of x_1, within which for each value of x we have

$$|f(x, y_0) - f(x, y_t)| < \sigma,$$

is as near $\Delta(x_1)$ as we may choose. In the same way, there exists a y_r, which may be equal to or different from y_t, such that the magnitude $\Delta'(x_1, y_r)$ of the neighborhood to the left of x_1, within which we have for each value of x

$$|f(x, y_0) - f(x, y_r)| < \sigma,$$

is as near $\Delta'(x_1)$ as we choose to make it.

On the line $y = y_0$, consider the sub-interval

$$\left(x_1 - \frac{\Delta'(x_1, y_r)}{2}, x_1 + \frac{\Delta(x_1, y_t)}{2}\right).$$

For every value x'' included within this interval, there exists the neighborhood (Fig. 2)

$$\left(x_1, x_1 + \frac{\Delta(x_1, y_t)}{2}\right)$$

or the neighborhood

$$\left(x_1 - \frac{\Delta'(x_1, y_r)}{2}, x_1\right),$$

according as x'' is to the right or to the left of x_1, such that for

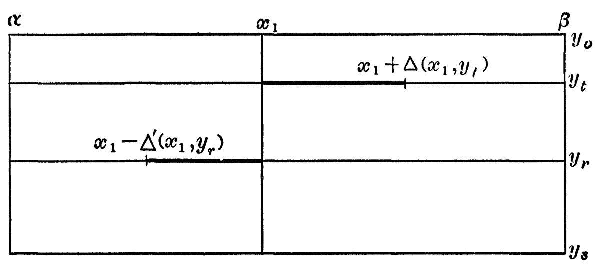

FIG. 2.

any value of x within these neighborhoods we have either

$$|f(x, y_0) - f(x, y_t)| < \sigma$$

or

$$|f(x, y_0) - f(x, y_r)| < \sigma.$$

Consequently, for every value x'' the corresponding value of $\Delta'(x) + \Delta(x)$ is greater than the smaller of the two numbers

$$\frac{\Delta'(x_1, y_r)}{2}, \quad \frac{\Delta(x_1, y_t)}{2}.$$

The lower limit of $\Delta'(x) + \Delta(x)$, namely l, must then be greater than or at least equal to the smaller of the same two numbers. But we have already seen that these numbers can be made to differ as little as we choose from

$$\frac{\Delta'(x_1)}{2}, \qquad \frac{\Delta(x_1)}{2},$$

respectively, and these latter are certainly greater than zero. Hence we have in any case

$$l > 0.$$

Since the lower limit of $\Delta'(x) + \Delta(x)$ for $\alpha \leqq x \leqq \beta$ is greater than zero, it can be shown that a finite number of segments of the lines lying between $y = y_0$ and $y = y_s$ can be found fulfilling the requirements of the theorem. The projections of these segments upon $y = y_0$ may, and in general will, overlap. That a finite number of them is sufficient in order that the sum of their projections shall completely fill the interval (α, β) may be shown by considering the end points of those parts which the projections contribute to the sum. If there are an infinite number of such points, then there must exist on the line

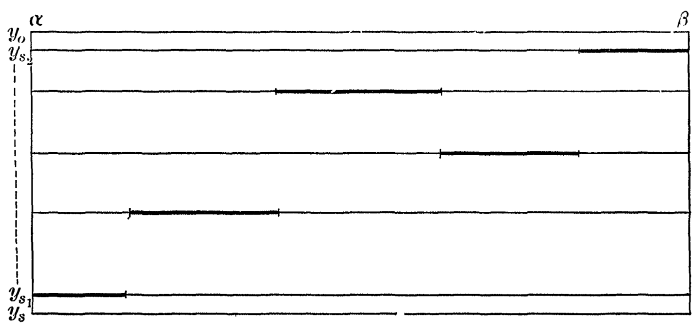

FIG. 3.

$y = y_0$ at least one limiting point of the corresponding values of x. This, however, can be avoided by the proper selection of the segments. For, let x_0 be such a limiting point. As we have seen, the value of $\Delta'(x) + \Delta(x)$ for $x = x_0$ is at least equal to l. Hence by the proper selection of a segment, we have for x_0 a neighborhood on $y = y_0$ equal to or greater than l which is entirely free from the end points mentioned above, and this is contrary to the supposition that they must of necessity become dense at some point.

It follows then that having selected an arbitrarily small positive number σ and a y_s at pleasure, we may choose a number $l > 0$ and lay off on the lines

$$y = y_{s_1}, y_{s_2}, \cdots y_{s_p},$$

$(y_{s_1}, y_{s_2}, \cdots y_{s_p})$ being taken between y_s and y_0, a finite number of segments fulfilling the requirements of the theorem. In case any segment extends beyond the given interval (α, β), we take only that portion of it which falls within (α, β).

The condition given by the theorem is also sufficient. To show this, let us suppose this condition fulfilled, and show that $f(x, y_0)$ is a continuous function for the interval $\alpha \leqq x \leqq \beta$. Let x_0 be any value of x within this interval and let y_s be any value chosen from the set (y) dense at y_0. If we assign to y_s any particular value, say y_{s_1}, then by hypothesis there exists a finite number of segments of lines lying between $y = y_{s_1}$ and $y = y_0$ fulfilling the condition of the theorem. Upon some one of these lines, say $y = y_{t_1}$, there exists for x_0 a neighborhood for which

$$|f(x, y_0) - f(x, y_{t_1})| < \sigma.$$

Again, if we put $y_s = y_{s_2}$, where y_{s_2} lies between y_{t_1} and y_0, there must exist a finite number of segments of lines lying between $y = y_{s_2}$ and $y = y_0$ likewise satisfying the condition of the theorem. Upon some one of these lines, say $y = y_{t_2}$, there exists for x_0 a neighborhood for which

$$|f(x, y_0) - f(x, y_{t_2})| < \sigma.$$

Continuing in the same manner, we obtain for x_0 an infinite succession of neighborhoods each point of which satisfies the inequality

$$|f(x, y_0) - f(x, y_{t_n})| < \sigma,$$

where $\underset{n=\infty}{\mathrm{L}}\, y_{t_n} = y_0$. These successive neighborhoods may vary in extent with y_{t_n}, but from some point on they are all different from zero. The condition of theorem I is therefore satisfied for the point $x = x_0$ and consequently $f(x, y_0)$ is continuous at that point. But x_0 was any point of the given interval $\alpha \leqq x \leqq \beta$, and hence $f(x, y_0)$ is continuous throughout the interval. With this, the demonstration is completed.

§ 4. As we saw in §1, the representation of a function by means of an infinite series of continuous functions may be regarded as a special case of the problem considered in §§ 2, 3. Theorem I, giving the necessary and sufficient condition that the limiting function shall be continuous for the value $x = x_0$ becomes the following for the case of infinite series.

I'. *Given an infinite series*

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

whose terms are continuous functions of x for the interval $\alpha \leqq x \leqq \beta$, and which converges for every value of x within the same interval. In order that the function $f(x)$ defined by this series be continuous at a point x_0 of the given interval, it is necessary and sufficient that for an arbitrarily small positive number σ and for every value of n greater than a sufficiently large number m there shall exist a neighborhood, which, however, may vary in extent with n, such that for every value of x within it we have

$$|f(x) - S_n(x)| \equiv |R_n(x)| < \sigma,$$

where $S_n(x)$ denotes the sum of the first n terms of the series and $R_n(x)$ is the remainder.

From theorem II we obtain the necessary and sufficient condition that $f(x)$ shall be continuous throughout the given interval $\alpha \leqq x \leqq \beta$. This condition may be stated for the infinite series as follows:

II'. *Given an infinite series of continuous functions which converges toward a limit $f(x)$ for each value of x within a definite interval $\alpha \leqq x \leqq \beta$. In order that $f(x)$ shall be continuous throughout this interval, it is necessary and sufficient that for an arbitrarily small positive number σ and for any integer m_1 there shall exist another integer $m_2 > m_1$ such that for some integer m lying between m_1 and m_2 we have*

$$|f(x) - S_m(x)| \equiv |R_m(x)| < \sigma, \quad \alpha \leqq x \leqq \beta,$$

where, however, m may change its value a finite number of times as x varies from α to β.

When we regard x and $1/n$ as the rectangular coördinates of a point in a plane, the above theorem may be stated more clearly perhaps as follows, where we make the same assumptions as before concerning the nature of the terms of the infinite series and its convergence.

II''. *An arbitrarily small positive number* σ *and an integer* m *being chosen entirely at pleasure, then in order that* $f(x)$, *defined as above, shall be continuous throughout the interval* $\alpha \leqq x \leqq \beta$, *it is necessary and sufficient that there shall exist a finite number of segments of the lines*

$$n = m + p_1, \quad m + p_2, \cdots, \quad m + p_r$$

fulfilling the following conditions: The sum of the projections of these segments upon the x-axis shall fill the entire interval $\alpha \leqq x \leqq \beta$, *and every point* $(x, 1/n)$ *of these segments shall satisfy the inequality*

$$|f(x) - S_n(x)| \equiv | R_n(x) | < \sigma.$$

Theorems II', II'' introduce a kind of convergence of infinite series which Arzelà has called uniform convergence by segments (convergenza uniforme a tratti), although sub-uniform convergence seems a more appropriate name.* It differs from ordinary uniform convergence in that for uniform convergence each segment must fill the entire interval in question, and, moreover, such a segment must be present for every value of n greater than some given integer. As we have seen, neither of these conditions has to be fulfilled for the case of sub-uniform convergence. In fact, a series might converge sub-uniformly in a given interval and yet not be uniformly convergent in any sub-division of that interval, however small.

On the other hand, sub-uniform convergence differs from the simple uniform convergence (einfach gleichmässige Convergenz) introduced by Dini.† While Dini's simple uniform convergence is like sub-uniform convergence in that it does not require that the segments mentioned in the theorem shall be present for all values of n from a certain point on, it differs from sub-uniform convergence in requiring that these segments shall each fill the entire interval in question. From these considerations it follows that when we have uniform convergence of a series in the ordinary sense, or simple uniform convergence in the Dini sense, the series converges sub-uniformly in the Arzelà sense. The converse, however, is not true. Since sub-uniform convergence is the necessary and sufficient condition for the continuity of

* See also Moore: BULLETIN, vol. 7, March, 1901, p. 257.

† See: Grundlagen, etc., p. 137.

the limiting function, it follows from what has been said that any series which converges uniformly or simply uniformly defines a continuous function, provided the terms of the series are also continuous functions within the given closed interval.

The distinction between these various kinds of convergence is illustrated by the following examples.

1. Let $f(x)$ be defined by the series

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \qquad 0 \leqq x \leqq \tfrac{1}{2}.$$

We have then

$$S_n(x) = \frac{x^n - 1}{x - 1}, \quad f(x) = \frac{1}{1 - x}, \quad |R_n(x)| = \frac{x^n}{x - 1}.$$

If now we select σ arbitrarily small, we can determine a value of n, say n_1, such that for $n > n_1$ we have

$$| R_n(x) | < \sigma$$

for all values of x for which the series is defined. In other words, if we consider any of the points on the lines given by putting

$$n = n_1, \quad n_1 + 1, \quad n_1 + 2, \cdots$$

the above inequality holds for all values of x within the given interval. The series converges uniformly therefore within this interval. We have also simple uniform convergence, since the projection of each of these segments upon the x-axis fills the entire interval $0 \leqq x \leqq \frac{1}{2}$ and the corresponding values of n from a set dense at infinity. Moreover, the series converges sub-uniformly; for it is at once evident that the conditions of theorem II'' are satisfied.

2. Given the series whose terms are formed in accordance with the following law :

$$u_{2m-1} = \frac{x}{mx^2 + (1 - mx)^2},$$

$$u_{2m} = \frac{-x}{(m + 1)x^2 + [1 - (m + 1)x]^2},$$

$$(m = 1, 2, 3, \cdots; \ -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}).$$

The series converges for all values of x within the given interval and gives

$$f(x) = \frac{x}{x^2 + 1 - x}.$$

When n is odd, say of the form $2m - 1$, we have

$$S_n(x) = \frac{x}{x^2 + (1 - x)^2}, \quad R_n(x) = 0.$$

Hence for all odd values of n, however large, we have

$$| R_n(x) | < \sigma.$$

On the other hand, when n is even, say of the form $2m - 2$, we have

$$S_n(x) = \frac{x}{x^2+(1-x)^2} - \frac{x}{mx^2+(1-mx)^2}, \; |R_n(x)| = \frac{x}{mx^2+(1-mx)^2}$$

As n takes increasing even values, it is impossible to find any integer beyond which for all such values of n, and for all values of x within the given interval, the inequality

$$|R_n(x)| < \sigma$$

shall exist when we choose $\sigma < 1$. This follows from the fact that the values of $|R_n(x)|$ approach 1 along the line $x = 1/m$ as n approaches infinity. Hence for $\sigma < 1$, those segments which fill the entire interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$, and for each point of which the above inequality is valid, can exist only for odd values of n. These values of n are, however, dense at infinity. Hence we have simple uniform convergence, but not ordinary uniform convergence. However, the conditions of theorem II$''$ are satisfied and the series converges sub-uniformly.

3. Given the series, the sum of whose first n terms is

$$S_n(x) = \frac{nx}{1 + n^2x^2}, \quad -1 \leqq x \leqq +1$$

This series converges for each value of x within this interval and gives

$$f(x) = \underset{n=\infty}{\mathrm{L}} S_n(x) = 0.$$

For any finite value of n, $S_n(x)$ at one point on the line $x = 1/n$ is equal to $\frac{1}{2}$ and at one point on the line $x = -1/n$ is equal to $-\frac{1}{2}$. Hence, for $\sigma < \frac{1}{2}$, there are no segments filling the entire interval $(-1, +1)$ such that for each point of them we have

$$|R_n(x)| < \sigma.$$

It follows that the series does not converge uniformly in the given interval, nor do we have simple uniform convergence. However, the series does converge sub-uniformly; for we may, for any arbitrary σ, select the required segments as follows.

First of all, consider the series for values of x in the neighborhood of $x = 0$. For $x = 0$, we have

$$f(0) = \mathop{\mathrm{L}}_{n=\infty} S_n(0) = 0.$$

Hence there exists a definite value of n, say $n = n_1$, such that

$$(1) \qquad |f(0) - S_{n_1}(0)| < \frac{\sigma}{3}.$$

Because $S_n(x)$ is continuous in x, there exists upon the line $n = n_1$ an interval $(0 - \delta_{n_1}, 0 + \delta'_{n_1})$ such that for every value of x within it, we have

$$(2) \qquad |S_{n_1}(0) - S_{n_1}(x)| < \frac{\sigma}{3}.$$

Moreover, since $f(x)$ equals zero for all values of x within the given interval $(-1, +1)$, we have for all values of x under consideration the inequality

$$(3) \qquad |f(x) - f(0)| < \frac{\sigma}{3}.$$

By combining the inequalities (1), (2), (3), we obtain the following relation

$$(4) \qquad |f(x) - S_1(x)| \equiv |R_{n_1}(x)| < \sigma,$$

which holds for all values of x within the interval $(0 - \delta_{n_1}, 0 + \delta'_{n_1})$ taken on the line $n = n_1$. Excluding this sub-interval from consideration, the given series converges uniformly throughout the remaining portion of the interval $(-1, +1)$. Hence, for the same value of σ, we can find on some line $n = n_2$, $n_2 > n_1$, a segment filling the entire remaining part of the given interval which lies to the right of $x = \delta_{n_1}$, such that at each point of this segment the inequality

$$| R_{n_2}(x) | < \sigma$$

is valid. For the same reason, we can find on some line $n = n_3$, $n_3 > n_1$, a segment filling the remaining part of the given interval to the left of the point $x = -\delta_{n_1}$, such that for each point of this segment the above inequality holds.

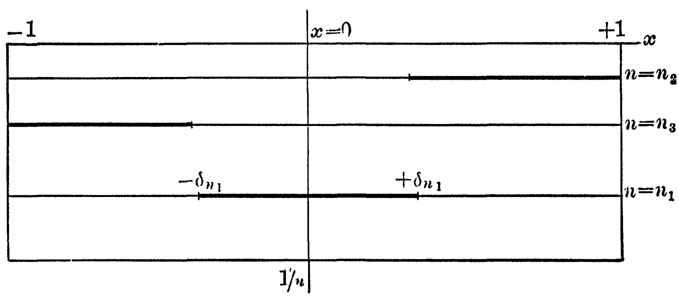

FIG. 4.

The three segments taken together satisfy the conditions of theorem II′, and hence the series converges sub-uniformly in the interval $-1 \leqq x \leqq 1$.

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