cartesian coördinates of a point on the mean surface of an isotropic congruence, where the functions f and f_1 which enter in these expressions refer to the minimal surface which is the envelope of the mean planes of the congruence. In the present case these functions refer to S_3, the adjoint minimal surface of S_2, the latter being the associate of S in the deformation determined by S_1. Hence we have the following theorem :

The middle envelope of an isotropic congruence is the adjoint of the minimal surface which is the associate surface in the infinitesimal deformation of the sphere, the directrix of the congruence.

PRINCETON,
November, 1901.

KRONECKER'S LECTURES ON THE THEORY OF NUMBERS.

Vorlesungen über Mathematik, von Leopold Kronecker, herausgegeben unter Mitwirkung einer von der Königlich Preussischen Akademie der Wissenschaften eingesetzten Kommission. In zwei Teilen. Zweiter Teil. *Vorlesungen über Allgemeine Arithmetik.* Bearbeitet und herausgegeben von DR. KURT HENSEL. Erster Abschnitt : *Vorlesungen über Zahlentheorie.* Erster Band. 8vo., xvi + 509 pp. Teubner, 1901.

IN the summer semester of 1841 Lejeune Dirichlet gave, for the first time, a course of lectures under the title of "Zahlentheorie" at the university of Berlin.* These lectures were attended by Kronecker. The subject was soon afterward added to the regular announcements not only at Berlin but also at the other German universities. The fact that during the winter semester of the current university year at least seven of the German universities offered courses on this subject, given by such well-known men as Frobenius, Weber, and Gordan, is sufficient evidence of the abiding interest in the theory which, according to Gauss, excels all other parts of pure mathematics in "its magic charm" and "its inexhaustible richness."

The volume before us is to be followed by a second on the same subject. In these volumes the editor aims to develop the theory of numbers in such a manner as to preserve the personal imprint of Kronecker, but to fill out the lacunæ which the lectures of this great arithmetician naturally pre-

* Under the title of "Anfangsgründe der höheren Arithmetik" Dirichlet offered a course on the theory of numbers at Berlin as early as 1833.

sent, especially since they were prepared for students who were generally familiar with the elements of the subject. In the present volume the editor carefully distinguishes his own contributions in the extensive list of "Anmerkungen" given at the end. The volume is divided into four parts, which bear the following headings: Divisibility and congruences of numbers, Domains of rationality and the theory of modular systems, Application of analysis to problems in the theory of numbers, General theory of residues of powers and proof of the theorem on the arithmetical progression.

The first three lectures are devoted to a historical introduction in which the chief contributions, beginning with the Greeks and ending with Dirichlet, Jacobi and Kummer, are sketched. In these introductory lectures one begins to feel that his guide is remarkably skilful in the choice of methods and takes a special interest in leading his hearers by the most direct routes to the questions which await solution. "Kronecker influenced the mathematical thinking of Germany as much through his lectures as through his published writings. He was a very stimulating and interesting lecturer. To an unusual degree he took his hearers into his confidence and allowed them the privilege of watching the actual evolution of his thoughts." *

Even in the first lecture attention is called to some interesting properties which await proof. In speaking about Euclid's proof of the fact that the number of primes is infinite, Kronecker remarks that it appears as if there were an infinite number of primes which differ from each other by the least possible number, viz., by two units. This property is included in the very difficult question in regard to the intervals in which prime numbers necessarily lie. Euclid's process gives intervals which increase too rapidly with the number. That there are indefinitely large intervals in which no prime number lies follows directly from the fact that the $n - 1$ consecutive numbers

$$n! + 2, \quad n! + 3, \quad n! + 4, \quad \cdots, \quad n! + n$$

do not include any prime, since $n! + i$ $(1 < i < n + 1)$ is divisible by i.

The law of the succession of prime numbers has been the object of many investigations. Among the various memoirs on this difficult subject, that presented in 1850 to the Academy of St. Petersburg by Tchebycheff is especially fa-

* Fine, "Kronecker and his arithmetical theory of the algebraic equation," BULLETIN, vol. 1 (1892), p. 183.

mous, since it proves the very useful theorem that there is always at least one prime between a and $2a - 2$ ($a > \frac{7}{2}$), a truth which had been conjectured by Bertrand at a much earlier date. In volume 4 of the *American Journal of Mathematics*, Sylvester gives some extensions of this memoir, which has been made more accessible through a reprint in Liouville, volume 17 (1852).

The first part of the present volume, Divisibility and congruences of numbers, begins with a study of the number concept. The definition of number which is given by Euclid and his followers is criticised as having no value for the matheematician. No definition may be considered justified until a method is supplied for determining in every concrete case whether the definition actually applies or not. A definition which does not stand this test is an artificial abstraction which has no place in mathematics. A definition of number must be such that the laws and fundamental operations of arithmetic can be naturally developed by means of it. This extremely practical tendency, as Kronecker himself called it, is one of the characteristic features of the volume.

The consideration of the nature of number and the fundamental operations is followed by a study of the divisibility of numbers, elementary congruences, the function $\varphi(m)$, and the theorems of Fermat and Wilson. The last lecture of this part is devoted to the invariants of a congruence. The function $f(x)$ is said to be an *invariant* of the congruence

$$k \equiv k' \bmod m$$

if it does not alter its value when x is replaced by any number which is congruent to it mod m. It is called a *characteristic* invariant if the congruence $k \equiv k' \bmod m$ is a direct consequence of the equation $f(k) = f(k')$. Only characteristic invariants are considered and it is observed that each of these invariants can be represented as a symmetric function of all the congruent numbers.

The second part begins with a study of congruences with respect to more than one modulus. If $m_1, m_2, \cdots, m_r$ are fixed numbers and if $c_1, c_2, \cdots, c_r$ assume all possible positive and negative integral values then the expression

$$c_1m_1 + c_2m_2 + \cdots + c_rm_r$$

will represent an infinite number of numbers. Each of these numbers is said to be divisible by the system $m_1, m_2, \cdots, m_r$. While the theory of these systems of divisors is practically superfluous when the numbers $m_1, m_2, \cdots, m_r$ are integers, it will be found very useful in the domain of integral functions of one or more indeterminate numbers. The interesting generalization of the concept of multiplication of numbers which is furnished by the composition of modular systems and the method of finding the highest common divisor by reducing a modular system to its simplest equivalent system are striking instances of the extension of common processes.

The developments of the theory of domains of rationality are of especial interest since this useful concept was practically introduced into mathematics by Kronecker. One of the most important results of this theory is the resolution of the modular system $(p, x^{p^n} - x)$ into irreducible factors whose degrees are divisors of n. The greater part of the developments are with respect to domains of rationality of one variable, but the last lecture of this second part is devoted to the modular systems in the domain of more than one variable. An elementary exposition of some of the fundamental concepts of this part is given by Professor Fine in the article cited above.

By the application of analysis to problems in the theory of numbers, Dirichlet obtained some very important results for which purely arithmetic proofs are still wanting. Following the systematic arithmetic of Gauss, the work of Dirichlet gave the second principal direction to the investigations in number theory in the nineteenth century. In the third part of the present volume, analysis is employed to establish some of the earlier results as well as to deduce a number of new and very important properties.

Some of the arithmetic functions, such as the totient of a number, exhibit such great irregularities that an analytic representation seems hopeless. However, it is sometimes possible to find mean values of such functions which give a remarkably clear insight into their general behavior. Among the mean values considered are those of $\varphi(n)$, $\frac{\varphi(n)}{n}$, and the number of divisors of a number. One lecture is devoted to the fundamental properties of cyclotomic numbers and the function $x^n - 1$. The series of Dirichlet is studied to a considerable extent and its usefulness is illustrated by means of some examples.

The main subjects of the fourth and last part are the theory of power residues and the theorem announced by Legendre that every arithmetical progression whose first term and common difference are prime to each other represents an infinite number of prime numbers. Dirichlet gave the first rigorous proof of this famous theorem, but this proof does not exhibit an interval in which a prime necessarily exists. In 1885 Kronecker succeeded in completing Dirichlet's proof in this important direction by showing that for each number m it is possible to determine a large number m' such that there is at least one prime, belonging to the arithmetic series in question, between m and m'. This instance serves to illustrate the high ideal which Kronecker set for himself and others in regard to completeness of proof.

It will be observed that the present volume does not enter upon the theory of forms, which occupies such a prominent place in the text-books on this subject. The elements of this theory are found in the works of Diophantus and his great commentator Fermat. At a later date Euler and Lagrange made a comprehensive study of the binary quadratic forms with respect to what numbers could be represented by them. Gauss was the first to study forms in general and to give a convenient notation for their further development. It is well known that a large part of the recent additions to number theory have been along this line.

Among the characteristic features of the present volume are the extensive use of congruences with respect to more than one modulus and the prominence of the concept of domain of rationality. Kronecker's work in this direction is so well known that it seems unnecessary to enter into details of explanations. It is perhaps sufficient to say that the editor has been remarkably successful in preserving the lecture style and thus he has given us a volume which forms unusually attractive reading, and, in places enters into details of criticism which are of especial interest to the beginner.

The extensive notes at the end of the volume greatly increase its value as they give not only the sources from which the material has been gathered and the additions of the editor, but also corrections and references. The present volume was delayed by the fact that just before his death Kronecker was engaged upon a study of the decomposition of modular systems which was left unfinished. The editor has thought it wise to complete these investigations and incorporate them in these lectures. As this work has been completed it is hoped that the second volume will follow very soon and that the inspiration which Kronecker's lectures gave to the study of the theory of numbers in Germany may thus be spread still more widely and become more permanent.

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