I have read with such pleasure since the days when I first met with Dr. Salmon's incomparable treatise on conic sections.

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NOTE ON PAGE'S ORDINARY DIFFERENTIAL EQUATIONS.

AN interesting review of this elementary text book was given by Professor Lovett in the BULLETIN, April, 1898. As the suggestions offered in the review cited are mainly of a general nature and appeal especially to those teachers familiar with the larger works of Lie, and hence able to make the desirable amplifications, it would seem worth while to address to the average reader or teacher of this text a few critical remarks of detailed character. Since my first acquaintance with Lie's groups and theories of integration, I have had the desire to introduce a class of mature students to the theory of ordinary and partial differential equations through the medium of continuous groups. Having used* the text by Page, I am more than ever convinced that the proper method (and one that will come more and more into vogue) of attacking differential equations is that which employs the powerful machinery—so simple when once mastered—set up and perfected by the illustrious Lie.

Being in full sympathy with the aims of the text, I was glad to find that, on the whole, the task had been well executed. I trust that in a second edition all objections that prove to be well grounded will be eradicated and that the errata, too numerous for an elementary text, will be corrected.

There is a curious mistake on p. 6, where the tangents to every integral curve of an ordinary differential equation are said to pass through the origin ! This is indeed the case for the only example given in the paragraph concerned. The answer to Ex. (19), p. 9, should be

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* During a year's graduate course in continuous groups, we devoted two months to the reading of Page's text, finding it a very practical supplement to a course of lectures on the general theory.

$$(x^2 + y^2)y'' - xy'^3 + yy'^2 - xy' + y = 0.$$

It would be well to establish the *vice versa* of p. 17, perhaps by use of formulæ (3), p. 11.

That the once extended (or m times extended) transformations of a one parameter group form a group seems to me unnecessary of proof. Indeed, a given transformation in x and y gives rise to definite transformations of the derivatives y', y'',$\cdots$ and it is entirely a matter of choice or convenience whether or not we will include in the *symbol* of the transformation not only the increments of x and y, but also the increments of y', y'',$\cdots$ If a formal proof be demanded, that proof should be as clear as possible. Instead of abridging slightly Lie's formal proof,* Page in his proof, p. 57, might well have amplified the derivation of (8) from (5), (6) and (7).

Examples (10) of p. 60 and (12) of p. 61 are quite faulty. The student is required to apply Art. 40, which is concerned with a family of ∞^1 curves whose equations may, therefore, be solved for the one arbitrary constant entering it. But the examples cited are concerned with families of ∞^2 curves. This difficulty may be obviated in Ex. (10) by considering separately the sub-families of ∞^1 conics in each of which the parameter b has any *particular* value. A *second* error in (10) lies in the incompatibility of the relations

$$\omega \equiv \frac{x^2}{a^2} \pm \frac{y^2}{b^2}, \quad U(\omega) \equiv x\frac{\partial \omega}{\partial x} \equiv \frac{2x^2}{a^2} = \Omega(\omega).$$

It is a pity that the proper historical setting was not given to the developments on pp. 69–71. In fact the investigation is identical with Lie's *first* (1869) method of integrating an ordinary differential equation of the first order admitting a known one parameter group. This earlier theorem proves that an ordinary differential equation admitting a known one parameter group *whose path curves are known* can be integrated by two quadratures. It is true that Page, carrying out the suggestion of Lie (l. c., p. 117), uses this investigation for the discussion of the problem to set up *all* differential equations of the first order which admit a given one parameter group. Although the developments given by Page really prove the integration theorem, no mention is made of the latter. He evidently prefers the later (1874) method given on p. 75.

* Lie-Scheffers, Vorlesungen über Differentialgleichungen, p. 267.

It is to be regretted that Page did not devote a little space to the consideration of ordinary differential equations admitting two essentially distinct infinitesimal transformations, especially as there results the simple and very important theorem that the quotient of the two resulting integrating factors is an integral of the given differential equation. The proof requires but a few lines (Lie-Scheffers, p. 124). The simple relation holding between two such infinitesimal transformations is readily deduced in a number of ways (Lie-Scheffers, pp. 125–132).

With reference to the table, pp. 96–97, I wish to call attention to the fact that one of the members of my course, Mr. Hathaway, has set up by general methods two very general types of infinitesimal transformations with the corresponding invariant differential equations. Most of the types given by Page are special cases of the following infinitesimal transformation involving three arbitrary functions:

$$- F(x) \frac{\partial f}{\partial x} + \{\varphi(x) F(x) y + \theta(x) F(x)\} \frac{\partial f}{\partial y},$$

leaving invariant the differential equation

$$\mu y' = \frac{1}{F} \omega(\mu y + \nu) - (\mu' y + \nu')$$

where ω denotes an arbitrary function of its argument and

where $$\mu \equiv e^{\int \phi(x) dx}, \quad \nu \equiv \int \mu \theta dx.$$

In this way Mr. Hathaway noted the error in the type (12) of p. 97, where xy' should read y'/x. Perhaps the error crept in by analogy to type (10).

We should have welcomed in Chapter V some examples of differential equations representing families of isothermal curves in addition to the two examples, viz., (1) p. 106 and (8) p. 107, taken from Lie-Scheffers.

The discussion in § 82 of differential equations of *degree* higher than the first might well be revised. The force of the word "rational," put in italics, is not clear; nor the reason for writing an integral in the form $y - \varphi(x, y, c)$ instead of the customary form $\varphi(x, y) - c$.

In the exposition in Chapter VII of Boole's treatment of Riccati's differential equations, Page uses as ultimate forms certain integrable differential equations in which the variables are not separate, whereas he might with equal ease have given forms with the variables separate. For example, why ask the reader to put the simple equation (3), at the bottom of page 120, into the cumbersome form (4) in order to integrate it? The remark on p. 120 that equation (2) is "much more easy to discuss" than equation (3) leads me to say that, in common with many others, I prefer to discuss the latter, but in the form to which it is easily reduced

$$(E) \qquad \frac{dy}{dx} + \tfrac{1}{2}y^2 = x^m.$$

This is Euler's special case of the reduced form of the *general* Ricatti's equation, the latter reduced form having an arbitrary function $\psi(x)$ in place of x^m in its right member. The cases in which (E) is integrable are found by using two simple types of substitutions, each transforming both the independent and the dependent variables. The discussion of the asymptotic case $m = -2$ would be of interest to the reader.

I do not understand the expression on p. 129 "loci composed of multiple points, cusps, etc." In order to speak of "the condition" on page 134, an inverse theorem would have to be established. The error of notation of using X_{m-1} for X_m occurs on pages 171, 175, 176 and 179. On p. 180, line 7, "the left member of" should be inserted after "in." At the bottom of p. 185, it is *proven* that the resulting equation is free from y, not that it is linear in v, a result sufficiently evident however.

A revision of §149 would be welcomed. Given X, Y, Z, the functions λ, μ, ν can *always* be found such that

$$(\mathrm{I}) \qquad \lambda X + \mu Y + \nu Z = 0\,;$$

indeed, if $X \neq 0$ for example, we may choose μ and ν arbitrary and solve for λ. But the resulting equation (3) would in general be sufficiently difficult to integrate. That an integral (3) is "obviously" an integral of (1) had to be proven to my class of able graduate men. We may give a simple proof as follows: We are given that

$$\frac{\partial \Omega}{\partial x}dx + \frac{\partial \Omega}{\partial y}dy + \frac{\partial \Omega}{\partial z}dz = 0$$

is a consequence of (3). Hence must

$$\lambda : \mu : \nu \equiv \frac{\partial \Omega}{\partial x} : \frac{\partial \Omega}{\partial y} : \frac{\partial \Omega}{\partial z}.$$

Then (I) shows that Ω is a solution of the partial differential equation

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0,$$

equivalent to the simultaneous system (1).

Evident misprints occur on p. 145, l. 7, p. 157, p. 182. It adds clearness to use y cot nx instead of cot nxy used p. 188.

A final remark is that it seems preferable to teach a general *method of procedure* for solving differential equations using freely transformations of the independent and dependent variables, rather that the application of a general formula. For example, the integration of the general linear differential equation of the first order is performed by a simple method, but by a complicated formula.

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TANNERY'S ARITHMETIC.

Leçons d'Arithmétique théorique et pratique. By JULES TANNERY. Paris, Colin et Cie, 1894. viii+509 pp.

THE present volume from the pen of the distinguished director of scientific studies at the École Normale Supérieure in Paris is the first work on arithmetic we have seen which while intended entirely for secondary instruction is written in accordance with the new ideas regarding the number concept and the need of rigor. It is thus a pioneer, perhaps even the inaugurator, of a revolution in secondary instruction in mathematics and as such will receive praise or censure according as the person in question is thoroughly awake to the crying necessity of reform in secondary mathematical instruction, or is not.

For fifty years or more slow changes have been taking place in the mathematical world. Their cumulative effect has completely transformed the aspect of mathematics from its bottommost foundations to the summit. Such mathematicians as Gauss, Cauchy, and Abel found the great structure of mathematics almost without foundation. Here is an extract of a letter of Abel to Hansteen, dated 1826 : "Je