$$F(z) = \sum_{n=0}^{\infty} a^n z^{n^2}, \qquad (|a| < 1),$$

is single-valued, provided $|a|$ is not too large.

The proof is as follows. Evidently

$$\left|\frac{f(z)-f(z')}{z-z'}\right| = \left|1 + \sum_{n=1}^{\infty} \frac{z^{a^n+1} + z^{a^n}z' + \cdots + z'^{a^n+1}}{(a^n+1)(a^n+2)}\right|$$

$$\geqq 1 - \sum_{n=1}^{\infty} \frac{|z|^{a^n+1} + |z|^{a^n}|z'| + \cdots + |z'|^{a^n+1}}{(a^n+1)(a^n+2)}$$

$$\geqq 1 - \sum_{n=1}^{\infty} \frac{1}{a^n+1} = 1 - \frac{1}{a+1} - \left(\frac{1}{a^2+1} + \frac{1}{a^3+1} + \cdots\right)$$

$$> 1 - \frac{1}{a+1} - \frac{1}{a(a-1)} > 0.$$

Hence $\qquad |f(z) - f(z')| > 0, \qquad$ q. e. d.

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NOTE ON THE PERIODIC DEVELOPMENTS OF THE EQUATION OF THE CENTER AND OF THE LOGARITHM OF THE RADIUS VECTOR.

BY PROFESSOR ALEXANDER S. CHESSIN.

IF we put with Professor S. Newcomb*

(1) $$E = ev_1 + e^2v_2 + e^3v_3 + \cdots$$

(2) $$\rho - \log a = e\rho_1 + e^2\rho_2 + e^3\rho_3 + \cdots$$

where E stands for the equation of the center and $\rho = \log r$, then v_i and ρ_i will be of the form

(3) $$iv_i = \tfrac{1}{2}\Sigma k_j^{(i)} \sin j\zeta,$$

(4) $$i\rho_i = \tfrac{1}{2}\Sigma h_j^{(i)} \cos j\zeta,$$

$$(j = i, \quad i-2, \quad i-4, \quad \cdots \quad, -i),$$

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* "Development of the perturbative function," Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac, vol. 5, part I., p. 12.

the coefficients $k_j^{(i)}$ and $h_j^{(i)}$ being rational numerical fractions subject to the conditions

$$k_j^{(i)} = -k_{-j}^{(i)}; \quad h_j^{(i)} = h_{-j}^{(i)}.$$

We propose to give in this note formulas by which these coefficients can be computed for any value of i and j.

If we put

$$E = \sum_{i=1}^{i=\infty} H_i \sin i\zeta, \tag{5}$$

$$\rho - \log a = \tfrac{1}{2}A_0 + \sum_{i=1}^{i=\infty} A_i \cos i\zeta, \tag{6}$$

then the comparison with formulas (1)-(4) gives

$$H_i = \sum_{m=0}^{m=\infty} \frac{k_i^{(i+2m)}}{i+2m} e^{i+2m}, \tag{7}$$

$$A_i = \sum_{m=0}^{m=\infty} \frac{h_i^{(i+2m)}}{i+2m} e^{i+2m}. \tag{8}$$

On the other hand it can be shown* that

$$H_i = \frac{2\sqrt{1-e^2}}{i} \sum_j \sum_q \frac{i^q}{q\,!} \left(\frac{e}{2}\right)^{j+q} N_{-i,j,q} \tag{9}$$

where j and q assume all integral positive values (zero included) such that

$$j + q = i, \quad i + 2, \quad i + 4, \cdots$$

If we develop $\sqrt{1-e^2}$ and put

$$H_i^{(2m)} = \sum_j \sum_q \frac{i^q}{q\,!} N_{-i,j,q} \qquad (-i + j + q = 2m), \tag{10}$$

then formula (9) becomes

$$H_i = \frac{2}{i}\left(\frac{e}{2}\right)^i \sum_{m=0}^{m=\infty} \left(\frac{e}{2}\right)^{2m} \Big[H_i^{(2m)} - 2H_i^{(2m-2)} - \cdots - \frac{1 \,.\, 3 \cdots (2m-3)}{m\,!} 2^m H_i^0 \Big]. \tag{11}$$

Comparing this formula with (7) we conclude that

* Tisserand: Mécanique Céleste, vol. 1, p. 243.

$$(12)\quad k_i^{(i+2m)} = \binom{i+2m}{i} \cdot \frac{1}{2^{i+2m-1}} \Big[H_i^{(2m)} - 2H_i^{(2m-2)} - \frac{1}{2!} 2^2 H_i^{(2m-4)} - \cdots - \frac{1.3 \cdots (2m-3)}{m!} 2^m H_i^0 \Big].$$

By this formula the computation of the coefficients $k_j^{(i)}$ is reduced to the computation of Cauchy's numbers for which the author has given a general formula.*

In order to obtain a similar expression for the coefficients $h_j^{(i)}$ we must first derive a development in powers of the eccentricity for the coefficients A_i. To this end we remark that

$$\frac{d\rho}{de} = \frac{d \log r}{de} = \frac{dr}{rde} = \frac{1}{e}\left(\frac{a}{r}\right) - \frac{1-e^2}{e}\left(\frac{a}{r}\right)^2.$$

On the other hand we have †

$$\frac{a}{r} = 1 + 2\sum_{i=1}^{i=\infty} J_i(ie) \cos i\zeta$$

$$\left(\frac{a}{r}\right)^2 = \frac{1}{\sqrt{1-e^2}} + \sum_{i=1}^{i=\infty} G_i^{(2)} \cos i\zeta$$

where $J_i(ie)$ is a Bessel's function and

$$(13)\quad G_i^{(2)} = 2\sum_j \sum_q \frac{i^q}{q!}\left(\frac{e}{2}\right)^{j+q} N_{i,j,q} \quad (j+q = i, i+2, i+4, \cdots).$$

Hence we may write that

$$\frac{d\rho}{de} = \frac{1-\sqrt{1-e^2}}{e} + \frac{1}{e}\sum_{i=1}^{i=\infty}\Big[2J_i(ie) - (1-e^2)G_i^{(2)}\Big]\cos i\zeta.$$

Now, it follows from (6) and (8) that

$$e\frac{d\rho}{de} = \tfrac{1}{2}e\frac{dA_0}{de} + \sum_{i=1}^{i=\infty}\sum_{m=0}^{m=\infty} e^{i+2m} h_i^{(i+2m)} \cos i\zeta$$

which, compared with the preceding formula, shows that

$$2J_i(ie) - (1-e^2)G_i^{(2)} = \sum_{m=0}^{m=\infty} e^{i+2m} h_i^{(i+2m)}$$

and we only need to find the coefficient of e^{i+2m} in the left hand side to obtain the required expression for $h_i^{(i+2m)}$.

* *Annals of Mathematics*, vol. 10, p. 1.

† Tisserand, Mécanique Céleste, vol. 1, pp. 224 and 242.

From (9) and (13) follows that

$$(1-e^2)\,G_i^{(2)} = i\sqrt{1-e^2}\,H_i$$

$$= 2\left(\frac{e}{2}\right)^i (1-e^2)\sum_{m=0}^{m=\infty} H_i^{(2m)}\left(\frac{e}{2}\right)^{2m}$$

$$= 2\left(\frac{e}{2}\right)^i \sum_{m=0}^{m=\infty}\left[H_i^{(2m)} - 4H_i^{(2m-2)}\right]\left(\frac{e}{2}\right)^{2m}$$

so that the coefficient of e^{i+2m} in $(1-e^2)\ G_i^{(2)}$ is found to be

$$\frac{1}{2^{i+2m-1}}\left[H_i^{(2m)} - 4H_i^{(2m-2)}\right]$$

while the coefficient of the same power of e in $2J_i(ie)$ is

$$(-1)^m\frac{1}{2^{i+2m-1}}\cdot\frac{i^{i+2m}}{m\,!\,(i+m)\,!}\cdot$$

Hence, we conclude that

$$(14)\quad h_i^{(i+2m)} = \frac{1}{2^{i+2m-1}}\left[4H_i^{(2m-i)} - H_i^{(2m)} + \frac{(-1)^m i^{i+2m}}{m\,!\,(i+m)\,!}\right]$$

which is the desired expression for the coefficients $h_j^{(i)}$.

To conclude we will express the coefficients $h_j^{(i)}$ by means of the $k_j^{(i)}$. To this end we multiply formula (7) by $\sqrt{1-e^2}$ and develop the right hand side in powers of e. Thus we obtain

$$\sqrt{1-e^2}H_i = \sum_{m=0}^{m=\infty} e^{i+2m}\left[\frac{k_i^{(i+2m)}}{i+2m} - \left(\frac{1}{2}\right)\frac{k_i^{(i+2m-2)}}{i+2m-2} - \cdots\right.$$

$$\left. - \frac{1.3\cdots(2m-3)}{m\,!}\left(\frac{1}{2}\right)^m\frac{k_i^i}{i}\right]$$

d, therefore,

$$h_i^{(i+2m)} = \frac{2(-1)^m\left(\frac{i}{2}\right)^{i+2m}}{m!(i+m)!} - \frac{ik_i^{(i+2m)}}{i+2m} + \frac{1}{2}\frac{ik_i^{(i+2m-2)}}{i+2m-2}$$

$$+\frac{1}{2!}\left(\frac{1}{2}\right)^2\frac{ik_i^{(i+2m-4)}}{i+2m-4}\cdots + \frac{1.3\cdots(2m-3)}{m!}\left(\frac{1}{2}\right)^m\frac{ik_i^i}{i}$$

which formula enables us to compute the values of the $h_j^{(i)}$ directly from the $k_j^{(i)}$.

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