Hence the most general infinitesimal point transformation which leaves the family of all concentric conics invariant is

$$Uf \equiv \left\{ ax(x^2+y^2) + \varkappa x + \frac{\lambda}{x} \right\} \frac{\partial f}{\partial x}$$

$$+ \left\{ ay(x^2+y^2) + \mu y + \frac{\nu}{y} \right\} \frac{\partial f}{\partial y}.$$

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A SOLUTION OF THE BIQUADRATIC BY BINOMIAL RESOLVENTS.

BY DR. GEORGE P. STARKWEATHER.

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THE solution of a given equation

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

consists in making it depend upon a series of resolvent equations

$$R_1 = 0, \quad R_2 = 0, \cdots$$

whose solution may be effected by known methods. Thus the general quadratic is reduced to a binomial $x^2 = a$, the cubic to a quadratic and a binomial cubic, and the biquadratic to a cubic and three quadratic equations. In all the solutions of the biquadratic the writer has seen these resolvent equations are not binomial, although Galois' theory shows us that they may so be taken in an infinite variety of ways, according to the particular system of resolvent functions chosen. In selecting such a system it is of course desirable to find one which will give as simple results as possible, and after some trial the set employed in the following lines seemed to be the best. It is hoped this solution will be of interest from two points of view: 1° as giving a *new* solution of the biquadratic in which the roots are given explicitly, *i. e.*, ready for calculation; 2° as affording an interesting application of Galois' methods.

Let the coefficients of

$$x^4 + bx^3 + cx^2 + dx + e = 0$$

be independent variables. For the domain of rationality R, consisting of these and an imaginary cube root of unity, α, the group of this equation is the symmetric group G_{24}. Adjoining the square root of the discriminant Δ the Galoisian group becomes the alternate group G_{12}. Here

$$\Delta = 4^4 (I^3 - 27J^2),$$

where

$$I = e - \frac{bd}{4} + \frac{c^2}{12},$$

$$J = \frac{ce}{6} + \frac{bcd}{48} - \frac{d^2}{16} - \frac{b^2e}{16} - \frac{c^3}{216}.$$

An invariant subgroup of G_{12} of index three is

$$G_4 = [1, (12)(34), (13)(24), (14)(23)].$$

Belonging to this is

$$\varphi_1 = x_1 x_2 + x_3 x_4,$$

which for the alternate group takes on the two other values

$$\varphi_2 = x_1x_3 + x_2x_4, \qquad \varphi_3 = x_1x_4 + x_2x_3.$$

Form

$$\psi_1 = \varphi_1 + \alpha\varphi_2 + \alpha^2\varphi_3.$$

We then know that

$$\psi^3 - \psi_1^3 = 0$$

must be a rational equation, and in fact

$$\psi_1^3 = D + \frac{3\sqrt{3}}{2}\sqrt{-\Delta},$$

where

$$D = \tfrac{1}{2} (2c^3 - 9bcd + 27d^2 + 27b^2e - 72ce).$$

After adjoining ψ_1 the group of $f(x) = 0$ becomes G_4. The values of φ_1 and φ_2 are, therefore, rationally known and we have

$$\varphi_1 = \frac{\psi_1 + \dfrac{H}{\psi_1} + c}{3}, \qquad \varphi_2 = \frac{\alpha^2\psi_1 + \alpha\dfrac{H}{\psi_1} + c}{3},$$

where

$$H = c^2 - 3bd + 12e.$$

The values of φ_1 and φ_2 are obtained from the three equations

$$\psi_1 = \varphi_1 + a\varphi_2 + a^2\varphi_3,$$

$$\frac{H}{\psi_1} = \varphi_1 + a^2\varphi_2 + a\varphi_3,$$

$$c = \varphi_1 + \varphi_2 + \varphi_3.$$

An invariant subgroup of G_4 of primeindex is

$$G_2 = [1, (12)(34)].$$

Belonging to this is

$$\varphi_1' = x_1 + x_2$$

which for G_4 takes on the other value

$$\varphi_2' = x_3 + x_4.$$

Form $$\psi_1' = \varphi_1' - \varphi_2'.$$

Then $$\psi'^2 - \psi_1'^2 = 0$$

must be a rational equation, and in fact

$$\psi_1'^2 = b^2 - 4c + 4\varphi_1.$$

Adjoin ψ_1' and the group becomes G_2.

G_2 has an invariant subgroup of prime index, the identical substitution, to which belongs

$$\varphi_1'' = x_1 + x_3.$$

For G_2 this takes on the other value

$$\varphi_2'' = x_2 + x_4.$$

As before, we form the equation

$$\psi''^2 - \psi_1''^2 = 0$$

where $$\psi_1'' = \varphi_1'' - \varphi_2''.$$

Then $$\psi_1''^2 = b^2 - 4c + 4\varphi_2.$$

Adjoining ψ_1'', the group becomes the identical substitution, and all rational functions of the roots are accordingly rationally known. In fact

$$x_1 = \frac{\psi_1' + \psi_1'' + \dfrac{K}{\psi_1'\psi_1''} - b}{4},$$

where

$$K = -b^3 + 4bc - 8d.$$

This is obtained from the four equations

$$x_1 + x_2 - x_3 - x_4 = \psi_1',$$

$$x_1 - x_2 + x_3 - x_4 = \psi_1'',$$

$$x_1 - x_2 - x_3 + x_4 = \frac{K}{\psi_1'\psi_1''},$$

$$x_1 + x_2 + x_3 + x_4 = -b.$$

x_1 is four-valued, for although ψ_1' is six-valued three values of it are present, namely, ψ_1', ψ_1'' and $\frac{K}{\psi_1'\psi_1''}$; and as the six values of ψ_1' are of the form $\pm A$, $\pm B$, $\pm C$, if we make $\frac{K}{\psi_1'\psi_1''}$ negative, or $-A$ (supposing K to be positive), we can only have

$$x_1 = \frac{-B + C - A - b}{4}$$

and

$$x_1 = \frac{B - C - A - b}{4};$$

while if we make $\frac{K}{\psi_1'\psi_1''}$ positive, or $+A$, we have

$$x_1 = \frac{B + C + A - b}{4}$$

and

$$x_1 = \frac{-B - C + A - b}{4}.$$

For convenience of reference the formulæ are appended in the order convenient for numerical application :

$$I = e - \frac{bd}{4} + \frac{c^2}{12},$$

$$J = \frac{ce}{6} + \frac{bcd}{48} - \frac{d^2}{16} - \frac{eb^2}{16} - \frac{c^3}{216},$$

$$\Delta = 4^4(I^3 - 27J^2),$$

$$D = \tfrac{1}{2}(2c^3 - 9bcd + 27d^2 + 27b^2e - 72ce),$$

$$\psi_1^3 = D + \frac{3\sqrt{3}}{2}\sqrt{-\Delta},$$

$$H = c^2 - 3bd + 12e,$$

$$\varphi_1 = \frac{\psi_1 + \dfrac{H}{\psi_1} + c}{3},$$

$$\varphi_2 = \frac{a^2\psi_1 + a\dfrac{H}{\psi_1} + c}{3},$$

$$\psi_1'^2 = b^2 - 4c + 4\varphi_1,$$

$$\psi_1''^2 = b^2 - 4c + 4\varphi_2,$$

$$K = -b^3 + 4bc - 8d,$$

$$x = \frac{\psi_1' + \psi_1'' + \dfrac{K}{\psi_1'\psi_1''} - b}{4}.$$

As an illustration let us apply these to the equation

$$x^4 - 2x^3 + x^2 + 2x - 2 = 0,$$

$$I = -\tfrac{11}{12}, \quad J = -\tfrac{37}{216}, \quad \Delta = -400, \quad D = 37,$$

$$\psi_1^3 = 88.96154, \quad \psi_1 = 4.46410.$$

(We take here the positive sign for $\sqrt{\Delta}$ and the real value of the cube root.)

$$H = -11, \quad \frac{H}{\psi_1} = -2.46410, \quad \varphi_1 = 1, \quad \varphi_2 = -2\sqrt{-1}$$

$$\left(\text{taking } a = \frac{-1+\sqrt{-3}}{2}\right)$$

$$\psi_1'^2 = 4, \quad \psi_1''^2 = -8\sqrt{-1}, \quad \psi_1' = \pm 2,$$

$$\psi_1'' = \pm(-2 + 2\sqrt{-1}), \quad K = -16,$$

$$x_1 = -1, \quad x_2 = 1 + \sqrt{-1}, \quad x_3 = 1, \quad x_4 = 1 - \sqrt{-1}.$$

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