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NOTE ON CONTACT TRANSFORMATIONS.

BY PROFESSOR EDGAR ODELL LOVETT.

IN a paper "Ueber die geodätische Krümmung der auf einer Fläche gezogenen Curven und ihre Aenderung bei beliebiger Transformation der Fläche" in the Zeitschrift für Mathematik und Physik, Vol. XXXVII., 1892, Dr. Mehmke, Schlömilch's successor as one of the editors of that journal, derives some interesting theorems relative to the change in the geodesic curvature of a curve on a surface when the latter is subjected to point transformations. Generalizations of these theorems are sought and are introduced by the following paragraph: "Die obigen Sätze erfahren eine Erweiterung und finden zugleich ihren natürlichen Abschluss beim Uebergange zu Transformationen, welche aus der Anwendung des Lie'schen Begriffes der Berührungstransformationen auf die Flächentheorie hervorgehen. Wir können jedem Linienelement auf einer gegebenen Fläche ein bestimmtes Linienelement auf einer anderen gegebenen Fläche so zuordnen, dass je zwei, im Sinne des Herrn Lie vereinigt liegenden Elementen wieder zwei vereinigt liegende Elemente entsprechen. Dann wird jede auf der ersten Fläche gezogene Curve sich in eine Curve (in besonderen Fällen auch in einen Punkt) der zweiten Fläche verwandeln und zwei sich berührende Curven werden im Allgemeinen in zwei sich ebenfalls berührende Curven übergehen.* Die punktweise Transformation einer Fläche ist hierin als besonderer Fall enthalten."

This is correct if the footnote and last sentence be omitted. These call for modification in order to correct misappre-

* "Derartige Transformationen von Flächen sind meines Wissens noch nicht untersucht worden. Sie lassen sich, wie leicht zu sehen ist, den räumlichen Berührungstransformationen keineswegs unterordnen."

hensions that the reader of the paper referred to may carry away with him. These transformations have been studied; they fall under Lie's categories of contact transformations; and for space the transformations in question are nothing but point transformations.

There are two cases to consider: 1° when the given surface is transformed into itself; 2° when it is transformed into another surface. In each case, by the geometrical definition of the transformations, two associated lineal elements are to be transformed into two associated lineal elements, that is, element association is to be transformed into element association.*

In the first of the above two cases there are two classes of transformations possible: (a) those transformations of the two dimensional manifoldness which transform lineal element association into lineal element association, these are the proper contact transformations of the surface; (b) those transformations of space which leave the surface invariant and at the same time transform lineal element association into lineal element association.

In the second case considered above there is but one category of transformations to study, namely, (c) those transformations of space which transform associations of lineal elements in space into lineal element associations.

Of these three classes of transformations those of the first class (a) are proper contact transformations of a two-dimensional manifoldness; and the transformations (b) are a particular type of the transformations (c). This third category of transformations has been studied by Lie and the precise nature of its transformations determined.

A lineal element of the plane is the ensemble of a point and a line through the point. Two elements infinitely near to each other are said to be associated when the point of the first lies upon the line of the second to terms of the second order. A family of ∞^1 lineal elements is said to form an element association when every element is associated with all the infinitely neighboring elements of the family. An element association of lineal elements in the plane consists of the ∞^1 lineal elements of a point or of those of a curve. Contact transformations of the plane are those transformations of lineal elements which change element

* Those who are not wholly familiar with these notions will find them explained in a review of Lie's Geometry of Contact Transformations which appeared in the BULLETIN, 2d series, volume 3, number 9, June, 1897, pp. 321–350. Proofs of several theorems quoted later will be found in the fifth section of the tenth chapter of the above work.

association into element association; those which transform point into curve are proper contact transformations; those which transform point into point are the so-called extended point transformations.

The notion lineal element in the plane has two correspondents in ordinary space: 1° the aggregate of a point and a line through it; 2° the ensemble of a point and a plane through it, the latter being called a surface element. Two lineal elements are said to be associated in space in the same sense as in the plane. Two infinitely neighboring surface elements are said to be associated if the point of one lies in the plane of the other to terms of the second order. Element associations of lineal elements and element associations of surface elements are defined in an obvious manner analogous to the lineal element associations of the plane. Similarly there are transformations of lineal elements of space which transform lineal element association into lineal element association, and transformations of surface elements which transform surface element associations into surface element associations.

Now it is a curious fact that the direct extension of the notion lineal element association to space avails but little, while the modified extension to that of surface element association proves to be most fruitful. Lie recognized very early in his geometrical work that transformations of surface element associations into such include transformations which are not point transformations (since there are five varieties of surface element associations, namely the ∞^2 elements of a point, the ∞^2 of a curve, the ∞^2 of a surface, the ∞^1 of an element band along a curve, and finally the ∞^1 of an element cone), namely the so-called proper contact transformations of space; while those transformations which change lineal element association into lineal element association are always extended point transformations. Then the above corrections are completely borne out if it be further observed that of these lineal element associations in space there are three varieties: 1° the ∞^1 elements of a curve; 2° the ∞^1 elements of an element cone; 3° the ∞^2 elements of a point.

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