THE FOURTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Constitution of the AMERICAN MATHEMATICAL SOCIETY provides for an Annual Meeting in the last week of December, at which the election of officers is held and other important business is transacted. On this occasion also an address is delivered by the President of the Society at the expiration of his first term of office. Coming in the holiday season, when most of the members of the Society are released from their professional duties, the Annual Meeting thus rivals the Summer Meeting in interest and attendance.

The Fourth Annual Meeting of the Society was held in New York on Wednesday, December 29, 1897. Nearly sixty persons were in attendance, including the following thirty-eight members of the Society: Professor M. Bôcher, Professor E. W. Brown, Professor A. S. Chessin, Dr. J. B. Chittenden, Professor J. E. Clark, Dr. A. Cohen, Professor F. N. Cole, Professor T. S. Fiske, Mr. G. B. Germann, Dr. W. A. Granville, Dr. G. W. Hill, Professor H. Jacoby, Mr. J. N. James, Mr. S. A. Joffe, Mr. C. J. Keyser, Professor P. Ladue, Professor P. A. Lambert, Professor G. Legras, Dr. E. O. Lovett, Dr. E. McClintock, Professor M. Merriman, Professor Simon Newcomb, Professor G. D. Olds, Dr. J. L. Patterson, Professor H. A. Peck, Mr. J. C. Pfister, Professor A. W. Phillips, Professor J. Pierpont, Professor M. I. Pupin, Professor J. K. Rees, Rev. G. M. Searle, Professor J. M. Taylor, Professor C. L. Thornburg, Professor J. M. Van Vleck, Professor J. B. Webb, Professor A. G. Webster, Professor F. S. Woods, Professor R. S. Woodward.

The meeting extended through two sessions, beginning at 10:30 a. m. and 2:30 p. m. The President of the Society, Professor Simon Newcomb, occupied the chair. The presidential address, delivered at the opening of the afternoon session, was entitled "The Philosophy of Hyperspace."

The Council announced the election of the following persons to membership in the Society: Mr. Joseph Allen, College of the City of New York; Professor Henry Benner, Albion College, Albion, Mich.; Mr. George B. Germann, Columbia University, New York, N. Y.; Professor A. C. McKay, McMaster University, Toronto, Canada; Dr. J. L. Mackay, Edinburgh Academy, Edinburgh, Scotland; Mr. Paul L. Saurel, College of the City of New York; Miss Mary Underhill, George School, Pa.; Mr. Alfred E. Western, London, England; Mr. Edmund T. Whittaker, Trinity College, Cambridge, England. Nine applications for membership were received. Reports were presented by the Treasurer and the Librarian. These reports are printed in the List of Members published by the Society in January.

At the annual election the following officers were chosen:

President,	Professor SIMON NEWCOMB,
First Vice-President,	Professor R. S. WOODWARD,
Second Vice-President,	Professor E. H. MOORE,
Secretary,	Professor F. N. COLE,
Treasurer,	Professor HAROLD JACOBY,
Librarian,	Professor POMEROY LADUE.

Committee of Publication,

Professor THOMAS S. FISKE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.

Members of the Council to serve until December, 1900,

Professor W. W. BEMAN,
Professor A. W. PHILLIPS,
Professor J. M. VAN VLECK.

The following papers were presented:

(1) Professor R. S. WOODWARD: "On the differential equations defining the Laplacian distribution of density, pressure, and acceleration of gravity in the earth."

(2) Professor M. BÔCHER: "The theorems of oscillation of Sturm and Klein."

(3) Professor M. BÔCHER: "On the roots of polynomials which satisfy certain linear differential equations of the second order."

(4) Professor A. S. CHESSIN: "On some points of the theory of functions."

(5) Professor SIMON NEWCOMB: Presidential address, "The philosophy of hyperspace."

(6) Dr. E. O. LOVETT: "Point transformations in elliptic coördinates of circles having double contact with a conic."

(7) Dr. E. O. LOVETT: "Certain invariants of a plane quadrangle by projective transformation."

(8) Mr. C. L. BOUTON: "Some examples of differential invariants."

(9) Dr. G. A. MILLER: "On the limit of transitivity of

the multiply transitive substitution groups that do not contain the alternating group.''

(10) Mr. C. J. KEYSER: ''Some theorems in n-dimensional space.''

(11) Dr. L. E. DICKSON: ''Orthogonal group in a Galois field.''

The presidential address appears in full in the present number of the BULLETIN, which also contains Dr. Dickson's paper. Dr. Miller's paper was published in the January number of the BULLETIN. Professor Bôcher's two papers and Mr. Bouton's paper will appear in later numbers.

Professor Woodward's paper presents an improved mathematical method for the treatment of the problem of the earth's density. Assuming the earth to be a sphere whose mass has attained its present arrangement in accordance with the laws of hydrostatics and gravitation, Laplace showed how by the aid of an hypothesis as to the compressibility of matter, the density, pressure and gravity at any point in the earth may be expressed as functions of the distance of that point from the earth's center. Since the time of Laplace this problem has been treated in an able manner by several writers;* and, so far as the physical questions involved are concerned, no improvement is needed. On the other hand, the methods used hitherto appear to be lacking in elegance and compactness, and is it chiefly to remedy these mathematical defects that the author has prepared the present paper.

Let V denote the potential, ρ the density, p the pressure, and g the acceleration of gravity at any distance r from the center of the sphere. Let these same symbols with the suffix zero denote the corresponding quantities at the surface of the sphere. Then for every point in the mass of the sphere Poisson's equation takes the form

$$\frac{\partial^2(rV)}{\partial r^2} + 4\pi k\,(r\rho) = 0, \tag{1}$$

where k is the gravitational constant.

The hydrostatic law gives

$$dp = \rho dV; \tag{2}$$

and Laplace's hypothesis as to the compressibility of matter is

$$dp = c\rho d\rho, \tag{3}$$

where c is a constant.

* See, for example, Thomson and Tait's Natural Philosophy.

Now ρ, p, V, and hence $g = -\partial V/\partial r$, are to be found from (1), (2), (3), taking into account the known values of the surface density, the mean density, and the surface pressure, along with the relation

$$g_0 = \tfrac{4}{3}\pi k r_0 \rho_m, \tag{4}$$

where ρ_m is the mean density of the earth.

The solution of the problem is shown to depend on the values of the products (rV) and $(r\rho)$. It is proved that each of these products satisfies the equation

$$\frac{\partial^4 Q}{\partial r^4} + a^2 \frac{\partial^2 Q}{\partial r^2} = 0, \tag{5}$$

wherein Q stands for either product, and a^2 is a constant; and that $(r\rho)$ also satisfies

$$\frac{\partial^2 Q}{\partial r^2} + a^2 Q = 0. \tag{6}$$

The characteristic features of the present paper consist in the use of equations (5) and (6) and in the entire avoidance of the volume integrals which appear in the usual treatment of the problem.

In Professor Chessin's paper which will be published in the *American Journal of Mathematics*, the author observes that G. Cantor's theory of irrational numbers presents a difficulty for a beginner in this study, namely, the conception of the number a priori defined by the regular sequence of which it is ultimately proved to be the limit. This pedagogical difficulty may be removed without introducing, as some authors do, the for a beginner equally difficult conception of Dedekind's theory.

The regular sequence is taken for the basis of the theory and is defined at the very beginning. Then sequences are considered satisfying the condition $a_k \leqq a_{k+1}$ or $a_k \geqq a_{k+1}$ and are called *normal* sequences, the first *ascending*, the second *descending*. It is readily shown that normal sequences are regular. An ascending normal sequence (a) and one descending (β) are said to form a *normal couple* if from a certain place onwards $0 < \beta_k - a_k < \varepsilon$. If a rational number A exists such that $a_k < A < \beta_k$ for all values of k, we say A belongs to the normal couple (a, β). It is readily shown that there can be but one such number. If such a rational number does not exist we create a new number and only one, which by definition is $> a_k$ and $< \beta_k$, and call it an *irrational* number. Thus, to every normal couple belongs one and only one number. It is proved next that with the terms of a given regular sequence we can form an infinite number of normal couples, but that *to all these couples belongs one and the same number*. Therefore, we say that *this number is defined by the given sequence.* It will be noticed that for this definition neither the conception of greater, smaller or equal irrational numbers, nor the definitions of sum or difference of such numbers are required, and yet the number defined by Cantor's sequence obtains a tangible form for a beginner. After this Cantor's method is followed closer. However, in the definitions of the sum, difference, product or quotient of two numbers defined by regular sequences a very important point is usually omitted, namely, the fact that *the results of the fundamental operations of arithmetic are independent of the manner in which the correspondence of terms of the two sequences may be established.* In fact, it is only in this respect that the numbers 0 and ∞ form an exception among the other members of the number system.

Another point of the theory of functions mentioned was the necessity of giving a more rigorously mathematical expression for the accumulations of discontinuities. The notion of a "reduced domain" was introduced and it was shown that the reduced domains had a limit independent of the manner of reduction. This limit it was proposed to name the linear or surface extension of the points of discontinuity, according as it was a linear or surface multiplicity of points.

Dr. Lovett's first paper will be published in the *American Journal of Mathematics.* If the equation of the conic be $x^2/a^2 + y^2/b^2 = 1$, and a system of elliptic coördinates be defined by the relations

$$x^2 + y^2 = \mu^2 + \nu^2 - c^2, \quad cx = \mu\nu,$$

the equation of the family of circles whose chord of contact is $x - x_1 = 0$ becomes

$$\mu^2 + \nu^2 - 2\mu\nu \cos\theta - a^2 \sin^2\theta = 0\,;$$

or in its equivalent form

$$\cos^{-1}\frac{\mu}{a} \pm \cos^{-1}\frac{\nu}{a} = \theta;$$

whence

$$\frac{d\mu}{\sqrt{a^2-\mu^2}} \pm \frac{d\nu}{\sqrt{a^2-\nu^2}} = 0 \equiv \Omega\,(\mu, \nu, \rho), \text{ where } \rho = \frac{d\nu}{d\mu}.$$

If
$$Uf \equiv \xi(\mu, \nu)\frac{\partial f}{\partial \mu} + \eta(\mu, \nu)\frac{\partial f}{\partial \nu}$$
be the infinitesimal transformation of the group of point transformations which leaves the above family of circles invariant, we have for its first extension
$$U'f \equiv \xi(\mu, \nu)\frac{\partial f}{\partial \mu} + \eta(\mu, \nu)\frac{\partial f}{\partial \nu}$$
$$+ \{\eta_\mu + (\eta_\nu - \xi_\mu)\rho - \xi_\nu \rho^2\}\frac{\partial f}{\partial \rho}.$$
The condition of invariance is $U'\Omega \equiv 0$; whence
$$\{\mu\xi \mp \xi_\nu \sqrt{(a^2 - \mu^2)(a^2 - \nu^2)}\}\rho^2 \pm (\eta_\nu - \xi_\mu)\rho$$
$$- \nu\eta \pm \eta_\mu \sqrt{(a^2 - \mu^2)(a^2 - \nu^2)} \equiv 0.$$
Hence
$$\frac{\mu\xi}{\xi_\nu} = \frac{\nu\eta}{\eta_\mu} = \pm\sqrt{(a^2 - \mu^2)(a^2 - \nu^2)}, \quad \xi_\mu = \eta_\nu.$$
The transformation is found finally in the form
$$Uf \equiv e^{\pm\psi(\mu,\nu)}\frac{\partial f}{\partial \mu} + e^{\pm\psi(\nu,\mu)}\frac{\partial f}{\partial \nu},$$
where
$$\psi(\mu, \nu) = \varphi(\mu)\sin^{-1}\frac{\nu}{a},$$
$$\varphi(\mu) = \sin^{-1}\frac{\mu}{a} \pm \log\{m[(\mu + \sqrt{a^2 - \mu^2})\sin^{-1}\frac{\mu}{a} - \mu]\}.$$

Similarly may be found the transformation leaving invariant 1° the family of circles whose chord of contact is $y - y_1 = 0$, 2° their orthogonal trajectories; finally those transforming one family into the other.

Dr. Lovett's second paper, which will be published in the *Annals of Mathematics*, is a contribution to the theory of a system of four coplanar points, and shows among other things how the group theory may be made to yield the details of elementary geometry. A geometrical configuration with m essential coördinates, has $m - r$ independent invariants by an r parameter Lie group. These $m - r$ invariants are solutions of the complete system of partial differential equations $\chi_1\varphi = 0, \cdots, \chi_r\varphi = 0$, where $\chi_1 f, \cdots, \chi_r f$ are the r independent infinitesimal transformations which generate the r parameter group. In the present paper theorems are obtained embodying invariant properties of a quadrangle by several projective groups. Incidentally there is found an interesting relation among the coördinates of four collinear points.

Mr. Keyser's paper may be summarized as follows: Specific dimensionality is not a definitive mark of any space, any space becoming n-dimensional on an always possible choice of element. For any assumed space the number of eligible non-linear elements is infinite, that of linear elements finite. The elements of both orders fall into pairs of reciprocals; for any pair there are always two equations each of which on dual interpretation defines both elements of the pair. The number,

$$-1 + (\nu + 1)(\nu + 2) \cdots (\nu + n) : n\,!$$

of essential constants in the general equation,

$$F(x_1, x_2, \cdots, x_{n+1}) = 0,$$

of degree ν, defining a locus (or envelope) of order (or class) ν in an n-flat, or linear space of n dimensions, the point (or $\overline{n-1}$-flat) being element, may be written

$$C^{*} = \nu n + \frac{\nu(\nu-1)}{1.2} \cdot \frac{n(n-1)}{1.2} + \frac{\nu(\nu-1)(\nu-2)}{1.2.3} \cdot \frac{n(n-1)(n-2)}{1.2.3} + \cdots;$$

from whose symmetry in ν and n follows: the maximal number of points (or $\overline{n-1}$-flats) arbitrarily assignable to a locus (or envelope) of order (or class, ν in an n-flat is the same as the number of points (or $\overline{\nu-1}$-flats) so assignable to a locus (or envelope) of order (or class) n in a ν-flat.

Arbitrarily to assign to a locus (or envelope) of order (or class) ν $C - m + 1$ points or $\overline{n-1}$-flats is equivalent, in respect to the determination of the form, to requiring it to belong to a family, $F + \sum_1^{m-1} \lambda_k F_k = 0$, of such forms going through (or touched by) an $(n-m)$-fold infinity of fixed points (or $\overline{n-1}$-flats), $m \underset{<}{\equiv} n$. If $m = n$, the forms of the

*Cf. Plücker: "Théorèmes généraux concernant les équations d'un degré quelconque entre un nombre quelconque d'inconnus." *Crelle*, vol. 16.

corresponding family have in common a *finite* number ν^n of points (or $\overline{n-1}$-flats). And all loci (or envelopes) of order (or class) ν having in common $C-n+1$ fixed points (or $\overline{n-1}$-flats) have also in common $\nu^n-C+n-1$ other fixed points (or $\overline{n-1}$-flats).

The maximum number C of *determining* elements is unchanged by the exchange of n and ν, but such is not in general true of the minimum number of extra *determined* elements; for the only positive integral solutions of the equation, $\nu^n-C+n-1=n^\nu-C+\nu-1$, are: (1) ν or $n=1$, n or $\nu=1, 2, 3, \cdots$; (2) ν or $n=2$, n or $\nu=3$. Case (2) yields interesting familiar theorems in 2- and 3-fold space. Excepting cases (1) and (2) there subsists the inequality, $\nu^n-C+n-1 \neq n^\nu-C+\nu-1$, where the right or the left member is the greater according as $\nu>n$ or $n>\nu$. When m is less than both n and ν, there is a corresponding inequality whose sense has the same criterion as above, though now the number of additional determined elements is infinite.

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THE EVANSTON MEETING OF THE CHICAGO SECTION.

The second meeting of the Chicago Section of the American Mathematical Society was held at Northwestern University, Evanston, on Thursday and Friday, December 30 and 31, 1897, Professor E. H. Moore, Vice-President of the Society, presiding.

The following members of the Society were present: Professor Henry Benner, Dr. E. M. Blake, Professor Oskar Bolza, Mr. A. C. Burnham, Professor Ellery W. Davis, Dr. L. W. Dowling, Dr. James W. Glover, Professor Arthur S. Hathaway, Professor Thomas F. Holgate, Mr. H. G. Keppel, Professor Joseph L. Markley, Professor Heinrich Maschke, Professor Malcolm McNeil, Professor E. H. Moore, Professor H. B. Newson, Professor J. B. Shaw, Dr. H. F. Stecker, Professor C. A. Waldo, Professor Henry S. White, Professor Mary F. Winston, Professor Alexander Ziwet.

The two days' session was fully occupied in reading the following papers: