ON THE COMMUTATOR GROUPS.

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THE operator $s^{-1}t^{-1}st$ has been called by Dedekind the commutator of s and t. When s and t are commutative this commutator is identity and *vice versa*. Since $s^{-1}t^{-1}st = (t^{-1}s^{-1}ts)^{-1}$, the inverse of a commutator is a commutator of the same two operators and may be obtained by merely interchanging the operators of the given commutator. Hence s and t have, in general, two commutators, the one being the inverse of the other. These two commutators are identical only when their order is 2 or 0.

When s and t represent successively all the operators of a group (G) their commutators generate a subgroup (G_1) of G. Since G_1 contains all the conjugates of any one of these generators multiplied into its inverse, these products being commutators of G, it must also contain all the given conjugates. Hence it is a self-conjugate (invariant) subgroup of G.

The quotient group of G with respect to G_1 is isomorphic to G and of order $g \div g_1$, g and g_1 being the orders of G and G_1 respectively. Since the commutators of the corresponding operators in these isomorphic groups must correspond, all those of the quotient group must be identity; that is, this quotient group is an Abelian or commutative group. If G has any other Abelian quotient group its self-conjugate subgroup which corresponds to identity in this quotient group must include G_1 since the commutator subgroup of an Abelian group is identity.

G_1 is a characteristic* subgroup of G, for the latter has only one commutator subgroup. If we represent the commutator subgroup of G_1 by G_2, that of G_2 by G_3, etc., it follows that each of the following groups

$$G, G_1, G_2, G_3, \cdots$$

is not only a characteristic subgroup of the one which immediately precedes it, but also of all those which precede it. Following Lie's notation, we may call $G_1, G_2, G_3, \cdots$ the first, second, third, $\cdots$ derived of G; the first derived and the commutator subgroup being equivalent terms.

*Cf. Frobenius, *Sitzungsberichte der Berliner Akademie*, 1895, p. 183.

Let G_α represent the first of these groups which satisfies the equation

$$G_\alpha = G_{\alpha+1}$$

It must then be identical with all those which follow it in the given series. We may therefore suppose that it terminates this series. If it is not identity it cannot be solvable, for its factors of composition must include composite numbers. In this case G is not solvable.* When $G_\alpha = 1$, G is evidently solvable.† Hence *the necessary and sufficient condition that a group is solvable is that we arrive at unity by forming its successive derived subgroups.*‡

When a group, like G_α, is identical with its commutator subgroup it has been called by Lie a perfect group. The necessary and sufficient condition that a group is perfect is that is not isomorphic to any Abelian group. Hence every simple group of a composite order must be perfect. The composite group formed by the product of any number of such simple groups is clearly also perfect.

Suppose that G' is the smallest self-conjugate subgroup of G which contains the commutators obtained by using for s and t every pair of non-commutative operators in any system of generators of G. The quotient group of G with respect to G' is generated by its operators that correspond to the given generators of G. The commutators of all these operators must therefore be equal to identity. Hence $G' = G_1$. As a group may generally be generated by a small number of its operators this is a convenient method to find the commutator subgroup of a given group.

For instance, if it is required to find the commutator subgroup of the symmetric group of order 24, we may use for s and t the two operators which may be represented by the substitutions abc and ad respectively. As these two substitutions generate the symmetric group their commutator and its conjugates must generate the commutator subgroup. From

$$s^{-1}t^{-1}st = acb.ad.abc.ad = adb$$

it follows that this subgroup is the alternating group of order 12.

While a group may have a large number of selfconjugate subgroups and even a large number of characteristic subgroups, yet it can have only one commutator subgroup. This is therefore not only a very special selfconjugate sub-

* Jordan : Traité des substitutions, p. 387.

† Cf. Ibid., p. 395.

‡ Cf. Lie : Continuierliche Gruppen, p. 548.

group, but it is also a special characteristic subgroup. It is therefore to be expected that we can prove theorems in regard to it which do not apply to the more general types of subgroups. Some of these have been incidentally noticed above. They may be summarized as follows:

THEOREM I. *When a group* (G) *is isomorphic to an Abelian group its commutator subgroup* (G_1) *is of a lower order than* G. *When this condition is not satisfied it is of the same order and hence identical to* G.

THEOREM II. *The quotient group of* G *with respect to* G_1 *is the largest Abelian group to which* G *has an* α, 1 *isomorphism.*

THEOREM III. *Every selfconjugate subgroup of* G *with respect to which it is isomorphic to an Abelian group must include* G_1.

THEOREM IV. G_1 *is a characteristic subgroup of* G; *i. e., it corresponds to itself in every simple isomorphism of* G *to itself.*

THEOREM V. *The necessary and sufficient condition that* G *is solvable is that we arrive at identity by forming the successive derived subgroups of* G.

THEOREM VI. G_1 *is the smallest selfconjugate subgroup of* G *which contains a commutator of each pair of non-commutative generators in any system of generating operators of* G.

To these theorems we shall add a few which seem to be more special.

THEOREM VII. *If a group of order* p^α *is not isomorphic to any Abelian group of order* p^3 *it contains only one selfconjugate subgroup of order* $p^{\alpha-2}$.

Since a group of order p^α contains at least one selfconjugate subgroup of order $p^\beta, \beta < \alpha$, the given group must contain at least one selfconjugate of order $p^{\alpha-2}$. With respect to this it is isomorphic to a group of order p^2. All groups of this order are Abelian. As the given group cannot be isomorphic to any Abelian group of a larger order its selfconjugate subgroup of order $p^{\alpha-2}$ must be its commutator subgroup. Hence it contains only one selfconjugate subgroup of this order.

COROLLARY I. *If a group of order* p^3 *contains more than one selfconjugate subgroup of order* p *it is Abelian.*

COROLLARY II. *Every selfconjugate subgroup whose order is obtained by dividing the order of a group by* p *or* p^2, p *being any prime number, includes the commutator subgroup of the group.*

THEOREM VIII. *If a commutator is commutative to one of its two operators its* α *power is the commutator of the* α *power of this operator and the first power of the other operator.*

First suppose that $s^{-1}t^{-1}st$ is commutative to t. Then

$$(s^{-1}t^{-1}st)^2 = s^{-1}t^{-1}st\,s^{-1}t^{-1}st = s^{-1}t^{-1}ss^{-1}t^{-1}st^2 = s^{-1}t^{-2}st^2$$

$$(s^{-1}t^{-1}st)^3 = s^{-1}t^{-2}st^2\,s^{-1}t^{-1}st = s^{-1}t^{-2}ss^{-1}t^{-1}st^3 = s^{-1}t^{-3}st^3$$

$$(s^{-1}t^{-1}st)^{\alpha}=s^{-1}t^{-\alpha+1}st^{\alpha-1}s^{-1}t^{-1}st=s^{-1}t^{-\alpha+1}ss^{-1}t^{-1}st^{\alpha}$$
$$=s^{-1}t^{-\alpha}st^{\alpha}.$$

When $s^{-1}t^{-1}st$ is commutative to s we obtain in a similar manner

$$(s^{-1}t^{-1}st)^{\beta}=s^{-\beta}t^{-1}s^{\beta}t.$$

Suppose that the operators of G are arranged in rows, each row containing all those that are conjugate to each other and no others ; *i. e.*, each row contains one and only one set of conjugate operators. When G is commutative each row consists of a single operator and *vice versa*. All the operators of a row may be obtained by multiplying one of them into certain operators of G. Since all of these last operators are commutators of G the order of G_1 cannot be less than the number of operators that are conjugate to any operator of G.

If we let s represent successively all the operators of a given row while t represents all those of G for each value of s, then will $s^{-1}t^{-1}st=1$ for g sets of values of s and t. Hence gk of the g^2 commutators of G are equivalent to identity, k being the number of the given rows. By writing after each operator all the different factors which make it equivalent to all the operators of the row in which it occurs, we obtain a system of factors which includes all the different commutators of G. All these factors are commutators and g of them are equal to identity.

It should be observed that the factors which occur after a given operator are conjugate to those which occur after any other operator of the same row. Hence we obtain at least one from each set of conjugate commutators by finding the factors into which we have to multiply one operator from each row in order to obtain all the other operators of the same row. Since the conjugates of a commutator are commutators of the same group, either all or none of the operators of a given row are commutators. This is another proof of the theorem that the commutator subgroup is selfconjugate.*

When one of the given rows contains the square of one of its operators, all its operators are evidently commutators. In general, if one of these rows contains both of the operators s_1 and s_1^{α} then will $s_1^{\alpha-1}$ and $s_1^{1-\alpha}$ be included among the given factors. Hence these must be commutators of G. From the equation

$$t_1^{-1}s_1t_1=s_1^{\alpha} \text{ or } t_1^{-1}s_1t_1s_1^{-1}=s_1^{\alpha-1}$$

* Frobenius, *Sitzungsberichte der Berliner Akademie*, 1896, p. 1348.

we have

$$(t^{-1}s_1t_1s_1^{-1})^{\beta} = t_1^{-1}s_1^{\beta}t_1s_1^{-\beta} = s_1^{(\alpha-1)\beta}$$

Under the given conditions all the powers of $s_1^{\alpha-1}$ must therefore be commutators.

Suppose that G is a transitive substitution group of degree p, p being any prime number. According to Sylow's theorem G contains $kp+1$ (k being some positive integer) conjugate subgroups of order p and its order is $ap\ (kp+1)$ (a being an integer >0). The number of elements in all its substitutions of order p is $p(p-1)\ (kp+1)$. This is the total number of elements in all the substitutions of G when $a=1$.* Hence a exceeds 1 whenever k exceeds 0.

The ap substitutions which transform one of the given subgroups of order p into itself form a group whose commutator subgroup is of order 0 or p as a is 1 or greater than 1. If we observe yet that the quotient group with respect to any selfconjugate subgroup of G, except identity, is cyclical and that all the substitutions of order p generate a simple selfconjugate subgroup of G, we obtain the following :

THEOREM IX. *The commutator subgroup of a transitive group of degree p is simple, and it includes all its substitutions of order p when the given transitive group is not regular.*

From this theorem we have directly that a transitive gronp of degree p is solvable when it contains only one subgroup of order p and that it is insolvable whenever it contains more than one such subgroup.

It may be well to add two important theorems in regard to the commutator groups which have been published within the last year, the one by Frobenius and the other by Dedekind. For the proof of these theorems we refer to their articles.

THEOREM X. *The number of linear factors of the group determinate of* G *is* $g \div g_1$.†

THEOREM XI. *When* G *is a Hamilton group its commutators are of order* 2 *or* 0.‡

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* Frobenius, *Crelle*, vol. 101, p. 287.

† *Sitzungsberichte der Berliner Akademie*, p. 1349.

‡ Dedekind, *Mathematische Annalen*, vol. 48, p. 557. The theorem given as a foot note in the same article, p. 553, was previously published, *Quarterly Journal of Mathematics*, vol. 28, p. 266.