BULLETIN OF THE

NEW YORK MATHEMATICAL SOCIETY

THE FOURTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Fourth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Toronto, Toronto, Canada, on Monday and Tuesday, August 16–17, 1897, thus immediately following the adjournment of the American Association for the Advancement of Science from Detroit to Toronto, and immediately preceding the Toronto Meeting of the British Association. Under these favorable conditions of scientific environment the success of the meeting was assured at the start. The actual outcome, however, exceeded all anticipation. In number and value of papers presented, in attendance of members and distinguished guests, and in scientific interest and enthusiasm, this meeting will rank among the most important scientific gatherings that have taken place on this continent, and is a distinct landmark of the vigorous growth of the Society and the scientific activity of its individual members.

Fifty-five persons were registered as in attendance, of whom forty-four were members of the Society. Twenty-one papers were read, all, with one exception, being by members of the Society. The list of those present is as follows: —

Professor Alfred Baker, Dr. Charlotte C. Barnum, Professor W. W. Beman, Dr. E. M. Blake, Professor F. H. Bigelow, Professor Oskar Bolza, Professor F. N. Cole, Professor A. T. DeLury, Dr. L. W. Dowling, Professor W. P. Durfee, Professor H. T. Eddy, Professor T. W. Edmondson, Professor Thomas S. Fiske, Professor A. R. Forsyth, Mr. J. C. Glashan, Dr. J. W. Glover, Professor A. G. Greenhill, Dr. Harris Hancock, Professor A. S. Hathaway, Professor Ellen Hayes, Professor O. Henrici, Dr. G. W. Hill, Professor T. F. Holgate, Dr. J. I. Hutchinson, Mr. H. G. Keppel, Professor J. E. Kershner, Professor P. A.

Lambert, Dr. G. H. Ling, Professor H. Maschke, Mr. W. McCabe, Dr. Emory McClintock, Mr. A. C. McKay, Mr. J. C. McLellan, Professor J. McMahon, Professor A. Macfarlane, Professor W. H. Metzler, Professor H. B. Newson, Professor Simon Newcomb, Mr. R. L. Sackett, Professor A. W. Scott, Mr. F. Shack, Professor J. B. Shaw, Mr. F. J. Smale, Dr. V. Snyder, Professor J. H. Tanner, Professor H. W. Tyler, Professor E. B. VanVleck, Professor C. A. Waldo, Professor A. G. Webster, Professor P. Wernicke, Mr. A. E. Western, Professor H. S. White, Mr. E. T. Whittaker, Professor R. S. Woodward, and Professor Alexander Ziwet.

Two sessions were held each day, at 10 A. M. and 2:30 P. M. The President, Professor Simon Newcomb, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. Herbert G. Keppel, Northwestern University, Evanston, Ill.; Mr. Cassius J. Keyser, Columbia University, New York, N. Y.; Professor William H. Maltbie, The Woman's College, Baltimore, Md. Three applications for membership were received. It was announced that the next Summer Meeting of the Society would be held at Boston, Mass., at or about the time of the meeting of the American Association for the Advancement of Science.

At the conclusion of the meeting resolutions were adopted tendering the thanks of the Society to the University of Toronto and its officers for many courtesies shown to the Society.

The following papers were read at the meeting:

(1) Dr. E. M. BLAKE: "Upon the representation by ruled surfaces of the curves drawn by mechanisms. Preliminary report, illustrated by models."

(2) Dr. L. W. DOWLING: "A contribution to the theory of plane curves."

(3) Professor ELLEN HAYES: "Note on the Folium of Descartes."

(4) Professor THOMAS F. HOLGATE: "A geometrical locus connected with a system of coaxal circles."

(5) Professor P. WERNICKE: "On the solution of the map-color problem."

(6) Professor H. B. NEWSON: "On the Riemann-Helmholtz-Lie problem of the foundations of geometry."

(7) Professor J. B. SHAW: "Quaternion invariantive operators."

(8) Dr. VIRGIL SNYDER: "The geometry of some differential expressions in hexaspherical coördinates."

(9) Professor E. B. VAN VLECK: "On certain differential equations of the second order allied to Hermite's equation."

(10) Professor OSKAR BOLZA: "Concerning the cubic involution and the cubic transformation of elliptic functions."

(11) Dr. J. C. FIELDS: "The determination of the rational function in the reduction of the general Abelian integral to the sum of a rational function and a fundamental system of elementary integrals."

(12) Dr. J. I. HUTCHINSON: "On the reduction of hyperelliptic functions ($p = 2$) to elliptic functions by a transformation of the second degree."

(13) Dr. EMORY MCCLINTOCK: "Further researches in the theory of quintic equations."

(14) Professor H. MASCHKE: "A theorem concerning the coefficients of linear substitution groups of finite order with n variables."

(15) Dr. G. A. MILLER: "On the commutator groups."

(16) Professor H. S. WHITE: "Collineations in a plane with invariant conic or cubic curves."

(17) Professor E. H. MOORE: "Concerning triple systems."

(18) Mr. F. W. FRANKLAND: "Theory of discrete manifolds."

(19) Mr. P. H. PHILBRICK: "The true transition curve."

(20) Dr. ARTEMAS MARTIN: "About sixth power numbers whose sum is a sixth power."

(21) Professor A. S. HATHAWAY: "Preliminary report on alternate functions of complex numbers."

The mechanisms considered in Dr. Blake's paper consist of a fixed plane m_0 with three coincident moving planes, m_1, m_2, m_3. To both m_0 and m_2 are connected both m_1 and m_2. The connections are of two kinds; first two planes may have a point in common about which they are free to rotate, or, second, they may have a straight line in common along which they are free to slide. If connections of the first kind only are used we have the well-known three-bar mechanism. Using both kinds of connections, there are ten possible types of mechanisms, but of which four are trivial. If to m_2 there be rigidly attached a straight line l, making with it an acute angle, then by the motion of m_2 due to the mechanism, l will describe a ruled surface, such that any section parallel to m_0 will be the curve drawn by a point of m_2 in the projection of l upon it.* A number of string models illustrating these surfaces were shown.

* A device similar to this was used by S. Roberts (*Proc. London Math. Soc.*, vol. 3, 1871) in a general investigation "On the motion of a plane under certain conditions."

In his paper on a contribution to the theory of plane curves, Dr. Dowling starts out from Zeuthen's article "*Sur les differentes formes des courbes planes du quatrième ordre*" (*Math. Ann.*, vol. 7, p. 410, 1874), where it was shown that a quartic curve necessarily possesses four real bitangents of the first kind. This result was extended by Professor Klein (Cf. *Math. Ann.*, vol. 10, p. 199) to curves of order n; viz., the following formula is true,

$$w' + 2t'' = n(n-2),$$

where w' is the number of real inflexions and t'' the number of isolated bitangents which the curve possesses. The object of this paper is to show, in the first place, that a curve of order n must necessarily possess $\frac{n(n-2)}{2}$, if n is even, or $\frac{n(n-2)-3}{2}$, if n odd, real bitangents of the first kind, that it may have more but cannot have fewer than this number. This is shown by considering the possible deformations which the curves of a sheaf of order n may undergo. In the second place, the equation of any curve of order n may be put in the form,

$$\alpha\beta\, U^{(0} - \varphi^2 \Pi = 0$$

in a finite number of ways, where $\alpha = 0$ and $\beta = 0$ are the equations of two real bitangents, $U^{(0)} = 0$ is the equation of a curve of order $n - 2$ possessing a node, $\varphi = 0$ is the equation of a conic, and $\Pi = 0$ is the combined equations of $n - 4$ straight lines. In the third place it follows from the above equation that a curve of order n may be regarded as the envelope of a special linear system of curves of order $n - 2$. The theory of poles and polars may be extended to the general curve of order n as has been done for the quartic. Cf. Salmon: *Higher Plane Curves*, p. 224.

Professor Hayes' paper on the Folium of Descartes contains a comparative study of a curve allied to the Folium, and an examination of a surface suggested by the two curves.

Professor Holgate discusses a system of coaxial circles in connection with a sheaf or pencil of rays. Let there be given in a plane a sheaf or pencil of rays through a point P and a coaxial system of circles having real intersections A and A'. Each ray of the sheaf is tangent to two circles of the coxial system and to each circle of the system two rays of the sheaf are tangent. Choosing then any circle of the system and one of its tangent rays, the second circle of the system to which this ray is tangent is immediately determined, then the second ray of the sheaf tangent to this second circle, and again a third circle tangent to this second ray and so on continuously. Restricting the first two circles to equality, and consequently the first ray to be parallel with the line of centres, it is found that the circuit of circles and rays will close with three circles and three rays if only the point P lies upon one of the parabolas having the line of centres for directrix and the point A or A' for focus. The centre of the third circle is determined by the straight line joining P and A or A'.

The general problem of Professor Wernicke's paper is immediately reducible to the following: To color, with four tints, a map in each vertex of which only three districts concur. If a map has been colored with four tints its frontiers may be marked with three indices so that concurrent frontiers have different indices. The converse of this is to be proven for surfaces of deficiency 0. Given a map correctly colored and with its frontiers marked, the author proves that any triangles, quadrangles, and pentagons can be introduced and correctly marked at the same time. The main theorem then follows by induction. Several corollaries are obtained, among them that any polyhedron in each summit of which three edges meet can by mere (triangular) truncation of single summits be changed into a polyhedron, every face of which has for the number of its sides a multiple of three.

Professor Newson discusses the Riemann-Helmholtz problem as stated by Lie, viz.: To find the properties which are common to the group of Euclidian and the two groups of non-Euclidian motion, and by which these three groups are distinguished from all other possible groups of motion. This paper treats only of the first part of the problem. The three groups of motions are defined with reference to the Cayleyan Absolute and all the real subgroups determined for each case. These subgroups are then classified with reference to the thirteen types of projective transformations in space. The types of motion corresponding to these types of projective transformations are illustrated by figures. The properties of the different types of motions are stated for the three kinds of space, and the corresponding types for the three spaces are compared. The properties common to the group of Euclidian and the two groups of non-Euclidian motions are thus found. The following results among others are reached: The most general type of motion in each kind of space is a screw motion, but the properties of these screw motions are somewhat different in the different spaces. Each kind of space admits of rotation as a special form of screw motion. Rotations in the three spaces are identical in their properties. Hyperbolic and elliptic space each have two types of translations. The properties of these translations are quite different. For a finite portion of space the elliptic, parabolic, and hyperbolic spaces are indistinguishable by their mechanical properties.

Professor Shaw's paper shows that the symbolic notation of Clebsch is simply an artificial form of a very natural notation for invariants, derived from quaternion expressions containing the operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

in the case of three variables (*i. e.*, ternary quantics) and similar expressions in the general case of n-ary quantics.

The differential expressions considered by Dr. Snyder are analogous to those studied by Klein in line geometry. When

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i}\right)^2 \equiv 0,$$

$f(x_1, \cdots x_6) = 0$ represents the ∞^3 spheres which touch its point locus. When

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i}\right)^2 \sum_{i=1}^{6}\left(\frac{\partial \varphi}{\partial x_i}\right)^2 = \Sigma\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial \varphi}{\partial x_i}\right),$$

the spheres common to $f = 0$, $\varphi = 0$ are the principal spheres of their focal surface; if f or φ is linear this surface reduces to a curve. When the combinant of three complexes vanishes their common spheres envelope a curve, instead of an annular surface. Applied to the quadratic complex, the system of confocal cyclides with its focal curves and lines of curvature is obtained without the use of point coördinates.

Hermite's differential equation,

$$\frac{d^2y}{du^2} \equiv \left[n\,(n+1)\,p\,(u) + h\right] y$$

can be thrown by the substitution

$$x = p(u) \quad \text{or} \quad u = \int \frac{dx}{2\sqrt{f(x)} = (x - e_1)\,(x - e_2)\,(x - e_3)}$$

into the form

$$f(x)\frac{d^2y}{dx^2}+\tfrac{1}{2}f'(x)\frac{dy}{dx}-\frac{n(n+1)x+h}{4}y=0.$$

As is well known, it admits of two solutions whose product is a polynomial in x. Other differential equations having the same or an analogous property have been given by Fuchs, Brioschi, Markoff, Lindemann, and G. W. Hill. Professor E. B. Van Vleck's paper gives a simple method of finding all regular linear differential equations of the second order which admit of two solutions whose product is a polynomial. Besides the cases given by the above authors, numerous others are derived. Also the group properties which characterized the equation are considered. The second part of the paper is devoted to an investigation of the roots of the polynomial when the differential equation contains four singular points. The investigation is analogous to that undertaken by Klein for Hermite's equation, and contains the results of the latter as special case.

Professor Bolza presented a proof of the following theorem: If two elliptic integrals of the first kind

$$\int\frac{(xdx)}{\sqrt{(xa)\,(xa_1)\,(xa_2)\,(xa_3)}}\quad\text{and}\quad\int\frac{(ydy)}{\sqrt{(yb)\,(yb_1)\,(yb_2)\,(yb_3)}}$$

are transformable into each other by a rational cubic transformation which may, without loss of generality, always be supposed in the polar form: $y_1\dfrac{\partial F}{\partial x_1}+y_2\dfrac{\partial F}{\partial x_2}=0$, F denoting a biquadratic in x_1, x_2, then *the roots* a, a_1, a_2, a_3 *and* b, b_1, b_2, b_3 *are the respective branch elements of two conjugate cubic involutions*, viz., the involution

$$\lambda_1\frac{\partial F}{\partial x_1}+\lambda_2\frac{\partial F}{\partial x_2}=0$$

and its conjugate.

Dr. Fields discusses elaborately the problem of the determination by a series of rational operations of the rational part of the general Abelian integral.

Dr. Hutchinson's paper on hyperelliptic functions is outlined by the following summary: Reduction of

$$\int F\left(z,\sqrt{f(z)}\right)dz\quad\left(f(z)=a_0z^6+a_1z^5+\cdots+a_6\right)$$

to sum of two elliptic integrals when a_0, a_1, $\cdots a_6$ satisfy the most general condition possible for a reduction involving a transformation of second degree. Study of a canonically dissected Riemann surface exhibiting the relation $\tau_{12} = \frac{1}{2}$ ($\tau_{12} = \tau_{21}$ being moduli of periodicity) and its (1, 2) correspondence with each of the two elliptic Riemann surfaces. Reduction of Klein's covariant integral $Q_{xy}^{\bar{x}\bar{y}}$ to corresponding elliptic integrals. Reduction of Klein's hyperelliptic σ-functions to the Weierstrass elliptic σ-functions. Derivation of the Borchardt and Weber equations of the Kummer surface in terms of hyperelliptic σ-functions. In case of reducibility, the Kummer surface becomes either (1) a doubly covered quadric surface, or (2) a Fresnel wave surface, or (3) an elliptic Kummer surface of the "fourth order."

In Dr. McClintock's paper on the quintic equations a method is suggested for detecting the rational factors of reducible quintics, applicable also to equations of other degrees; also a method for recognizing quintics having just two imaginary roots, and therefore not resolvable. The author's former paper of 1885 (*Amer. Journal*, vol. 8), largely reproduced by Cayley (Collected Papers, end of vol. 4.), is recalled, including his new resolvent in t, or more generally in τ, as well as his form in v of the older (Malfatti's) resolvent; also his recognition and introduction of these rational quantities t, v, s, such that $s = t^2v$, two of which are required in any resolution, and his formulæ for expressing the elements, u_1, u_2, u_3, u_4, and therefore the roots, in terms of t, v, and the coefficients. He now derives the same results (one of the formula being at the same time simplified) directly, without algebraic process, by a method applicable also to other equations, and precisely the reverse of that of Bezout and Euler, who define the elements as functions of the roots; he defines all his quantities, including the unknown quantity, as functions of the elements, being thus enabled to prove all his formulæ by immediate substitutions involving no irrational quantity. He next proves that all resolvable quintics depend upon four rational parameters, q, r, t, w, by assigning values to which resolvable quintics may be constructed at will, viz., the quintic being $y^5 + 10\gamma y^3 + 10\delta y^2 + 5ey + \xi = 0$, he makes $\gamma = ql$, $\delta = (r + t)l$, $v = l^2 \div (1 + w^2)$, where

$$l = [t^2 w^2 - r^2 (1 + w^2)] \div 4(q + 1)[q^2 (1 + w^2) - 1];$$

after which

$$e = \gamma^2 + v(3 + 4w) + (\gamma^2 - v)^{-1}(\gamma\delta^2 - \gamma t^2 v - 2\delta tvw),$$

$$\xi = (\gamma^2 - v)^{-1}(25tv^2 - \gamma t^3 v + \delta t^2 v - etv - 10\gamma^2 tv + \gamma^4 t - \gamma^2 et + \gamma\delta^2 t + 2\gamma\delta e - \delta^3).$$

He modifies this system for various critical cases, gives reasons why simpler parameters cannot be devised, notes that there are four quintics for which the value of t is the same and that of v is the same, and refers to earlier partial solutions, namely, for the trinomial $y^5 + 5ey + \xi = 0$, of the problem of assigning the necessary form of the coefficients of resolvable quintics. He proves finally, with comments, that if his sextic resolvent in τ be reduced to a quintic by dividing out the rational root, the resolvent in τ of the new quintic must have the same rational root.

Professor Maschke established the theorem that the coefficients of a linear homogeneous substitution group of finite period with any number of variables are cyclotomic, *i. e.*, are rationally expressible in terms of roots of unity.

Dr. Miller discusses certain properties of the commutator groups. The operation $sts^{-1}t^{-1}$ is said to be the commutator of the two operations s and t. When s and t are commutative their commutator is identity and, *vice versa*. From the equation $sts^{-1}t^{-1} = (tst^{-1}s^{-1})^{-1}$ we see that the inverse of a commutator may be obtained by interchanging its two operators, and from the equations

$$l^{-1}sts^{-1}t^{-1}l = l^{-1}sts^{-1}ll^{-1}t^{-1}l = S_1tS_1^{-1}l^{-1}t^{-1}l = Sl^{-1}tlS^{-1}l^{-1}t^{-1}l = STS^{-1}T^{-1}$$

it follows that the transforms of the commutator of s and t with respect to the operations of a group that includes these operators are commutators of the same group. The subgroup generated by all the commutators of a given group is self-conjugate. It is the smallest subgroup with respect to which the given group is isomorphic to a commutative group. When the given group is simple this subgroup is identical to it, but the converse is not true. The subgroup generated by the commutators of the given self-conjugate subgroup is also self-conjugate in the entire group. The necessary and sufficient condition that a group is solvable is that we arrive at identity by finding the subgroup generated by the commutators of a group, then the subgroup generated by the commutators of this subgroup, etc. If we let s and t represent successively all the operations of a group $sts^{-1}t^{-1} = 1$ gk times, g being the order of the group and k the number of its systems of conjugate operations. *References.*— *Quarterly Journal*, vol. 28, p. 266 (1896). Frobenius, *Berliner Sitzungsberichte*, December, 1896. Dedekind, *Mathematische Annalen*, January, 1897.

Collineations with invariant conic have eight free constants. The invariant condition determining the ninth has been found by seeking *Bahn-curven* belonging to iterations of a collineation, and specializing their parameters to reduce them to conics. Professor White attacks the problem directly, postulating at once an invariant conic which is not degenerate. After this a similar method gives corresponding conditions for invariant cubic. Both problems are elucidated and in part solved by geometric schemes.

Professor Moore's paper is a continuation of the author's investigations on triple systems. A triple system in t letters is regular if it is invariant under a regular substitution group of order t on its t letters. An analysis is given of the general regular system and explicit construction of a system in $t = 6m + 3$ letters, regular with respect to any regular group of order $t = 6m + 3$, which contains a self-conjugate element of period 3, and a group of order $2m+1$ not containing that element. Professor Moore's paper appears in full in the present number of the BULLETIN.

Mr. Frankland's papers on the theory of discrete manifolds have been privately printed from time to time, the series now extending to Part IX. These papers contain a discussion of the notion of continuity, certain alternative implications of this notion as applied to space, the connection between the geometry of continuous manifolds and the tactical arrangement of units in a discrete manifold, "facts of nextness" and "microcycles or meshes or circuits" in a discrete manifold, hypotheses respecting the geometry of minute regions, suggesting the possibility of an explanation of mass, space occupancy and motion in terms of time and tactical arrangement only, discussion of certain principles which must underlie a suitable notation for theorems relating to discrete manifolds, and conditions under which the units of a discrete manifold constitute (a) an enumerable, and (b) a denumerable mass.

Mr. Philbrick's paper, which is to be published as a treatise, contains an elaborate discussion of a specially simple form of transition curve together with an historical note on this subject.

Dr. Martin presented a number of interesting arithmetical identities concerning sixth power numbers.

Professor Hathaway defines a function of one or more complex number arguments which is linear and homogeneous in each argument and is reversed in sign by the interchange of any two of its arguments as a linear alternate or simply an alternate. The author proposes to investigate the geometric relations of quaternion and higher complex numbers, and to determine the relations that exist between alternate identities and integrations through a given space and over the boundary of that space. An instance of such a connection has been already discussed by the author in the Proceedings of the Indiana Academy of Science, 1891.

A portion of the Tuesday afternoon session was devoted to a general discussion of the following topics: (1) *The accurate definition of the subject matter of mathematics;* (2) *The vocabulary of mathematics, the possibility of correcting and enriching it by coöperative action.*

F. N. COLE.

COLUMBIA UNIVERSITY.

CONCERNING REGULAR TRIPLE SYSTEMS.

BY PROFESSOR ELIAKIM HASTINGS MOORE.

Read before the AMERICAN MATHEMATICAL SOCIETY at its Fourth Summer Meeting, Toronto, Canada, August 17, 1897.

§ 1.

Introduction. Definitions and notations.

$$\text{A} \qquad \begin{array}{ll} k\text{-ad} & [l_1, \cdots, l_k] \\ k\text{-id} & \{l_1, \cdots, l_k\} \end{array}$$

is an arrangement of k letters or elements $l_1 \cdots l_k$ in which the order $\begin{array}{c}\text{is not}\\ \text{is}\end{array}$ material. The letters are distinct.

A *triple system* Δ_t is an arrangement of t letters in 3-adic triples in such a way that every 2-adic pair appears exactly once in some triple of the triple system. There is no question of order of the triples of the triple system. t must have the form $t = 6m + 1$ or $t = 6m + 3$; t denotes always a number of such (say) *triple form.*

A triple system Δ_t is invariant under a certain (largest) substitution group G^t on its t elements; the Δ_t and the G^t belong each to the other; the G^t is a *triple group.*