all time. The examination of the idea of chance shows the the latter presupposes an infinitely refined agnosticism, the limit of all possible or conceivable increase of knowledge. The theory of the Unknowable, the Absolute, the Will and wills were cognized under this philosophical aspect.

The object of Dr. Martin's paper was to deduce from the ordinary logarithmic series by proper modifications rapidly converging series from which the number corresponding to any given logarithm may be computed without tables. Four forms of series were given available under different conditions.

Dr. R. W. Willson and Professor B. O. Peirce presented a table giving the first forty roots of the Bessel equation $J_0(x) = 0$ and the corresponding values of $J_1(x)$. The first ten values of x for which $J_0(x)$ vanishes have been given to ten places of decimals by Meissel. The next thirty and the values of $J_1(x)$ corresponding to the first forty roots have been computed by the authors by means of Vega's ten-place tables of logarithms, except in the few cases where a greater number of places was necessary, and then recourse was had to Thoman's tables. The computation has been gone over twice.

Professor Taber's paper contains a theory of the special linear homogeneous group in n variables constituting a generalization of Study's theory of such a group for $n=2$. The results show that there are as many species of transformations of the special linear homogeneous group in n variables as there are factors of n. The paper also contains a theory of certain other sub-groups of the general projective group, analogous to Study's theory of the special linear homogeneous group of the plane.

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CELESTIAL MECHANICS.

Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac. Vols. V., VI., VII. Washington, 1894–5.

The progress made in modern times toward the practical solution of the problems of celestial mechanics has been in no way behind that of the more theoretical investigations. New and more efficient methods of attacking the problems have been invented, greater rigour has been introduced and numerical results have been carried to a much higher degree of approximation than was even deemed possible in earlier days. The determinations of the constants from the large masses of scientific observations which are now available, extending over some two hundred years, have enabled astronomers to obtain data for the formation of numerical tables to represent the motions of the planets and satellites with an accuracy far beyond that originally required. This refined accuracy is, of course, one of scientific interest, and it may almost be said to have but one object in view—the verification of the Newtonian law of gravitation; at the present time the observed motions are referred to this cause alone. That a single law of this exact nature is sufficient to account for the complicated phenomena of the solar system, must be regarded as very remarkable in view of the approximate nature of most of the assumptions made for the grouping of other observed phenomena. The unexplained differences between the places of the heavenly bodies, as calculated by means of the Newtonian law and as deduced from observation, small as they are, are nevertheless, in some cases, well defined, and they appear to show the necessity either of some modification of the law of the inverse square or of the introduction of some unknown cause of disturbance or, perhaps, of the existence of some new force. The method adopted by Professor Newcomb in order to take account of these differences will be mentioned below.

It might seem at first sight that many suppositions might be made to account for these deviations, but a little reflection will show that this is not necessarily the case. A supposition may be made in order to explain certain motions of one or two bodies, unaccounted for by the Newtonian law, and its magnitude and general effect may be calculated. But, owing to the large number of bodies in the solar system, it is usually possible to find at least one on which the effect is much greater than the observations will admit, and the supposition must therefore be rejected. It will be seen, then, that the difficulty is not that of choosing out one of a number of possible causes of disturbance, but rather that of discovering one which will satisfy the observed phenomena in the motions under consideration, and at the same time not give rise to other phenomena which we know, from observation, cannot exist. If an excuse were needed for the refinements to which the numerical solutions of these problems are subjected, a sufficient one is contained in the remarks just made. The Newtonian law of gravitation is so wide in its application to all problems of physics that no efforts made towards a verification of its accuracy and a determination of its limitations will be useless.

In the volumes before us the American Nautical Almanac office has given a further proof of the efficiency and activity which has characterized it during the last twenty years. In a prefatory note to the first volume of the series, published in 1882, Professor Newcomb announced that 'the objects of the series of papers ...... are, a systematic determination of the constants of astronomy from the best existing data, a reinvestigation of the theories of the celestial motions, and the preparation of tables, formulæ and precepts for the construction of ephemerides and for other applications of the results.' A brief glance at the contents of the seven volumes published up to the present date will show how fully these objects have been carried out, and the names attached to the various memoirs will be a sufficient warrant for the permanent value of the results. Since the time of Leverrier it may be safely said that no government office has published astronomical results of such value, whether we view them from the side of pure science or in the more practical direction of the construction of ephemerides and tables. Most of the first four volumes, which I do not intend to discuss here, are concerned with the theoretical investigations of the celestial motions and with the determinations of the fundamental constants of astronomy from the existing data. The measurement of time, the velocity of light, catalogues of stars and the discussion of ancient eclipses have all received attention as necessary preliminaries to the accurate determination of the constants which refer more particularly to the motions of planets and satellites. In the later volumes we are given investigations of the perturbations of the planets, both periodic and secular, by new methods or by better and more complete applications of existing methods; the reduction of thousands of observations for the determinations of the elements; and the discussion and adjustment of these values so as to best represent the planetary motions. The last step is the formation of tables obtained from the results of theory alone, without those empirical corrections (with one exception), which the observations require, but which the theory does not seem to suggest.

The final stages of the work are now beginning to appear. The three volumes published during the last eighteen months contain, besides some of the concluding theoretical investigations, new tables for the calculations of the positions of the Sun, Mercury, Venus, Jupiter and Saturn. For convenience, I shall put the memoirs into three divisions:

I. Papers on the Planetary Theory (Vol. V.).
II. Tables of the Planets (Vols. VI., VII.).
III. A Paper on the Lunar Theory (Vol. V.).

I. Of the papers on the planetary theory by Professor Newcomb, the first is entitled, *A Development of the Perturbative Function in cosines of multiples of the Mean Anomalies and of angles between the perihelia and common node and in powers of the Eccentricity and Mutual Inclination.* The main feature of the method of development is the use of symbolic operators, as explained in an earlier paper* by the same writer. In Vol. III. of the series it was applied to the development of the perturbative function in cosines and sines of multiples of the *eccentric* anomalies of the two planets. In general, however, the methods used for the determination of the planetary perturbations demand that the expansions be made according to multiples of the *mean* anomalies, and we must therefore obtain the values of the operators in the latter case. It is not necessary to state here the method of procedure; it rests mainly on the theorem, proved by Professor Newcomb in the papers referred to above, that the coefficients for any powers of the eccentricities e, e' can be found, when those for $e = 0$ and for $e' = 0$ have been found, by a simple multiplication of the resulting operators. The main part of the memoir is taken up with the algebraical values of the operators. The expressions are carried to the eighth order with respect to the eccentricities for the terms which are independent of the inclinations, and to the seventh order for the terms which depend on the square of the inclination. The continuation of the process, for any degree of accuracy with respect to powers of the eccentricities and inclination, is cast into the form of precepts for computation. It may be stated that the indeterminates which enter into each operator are, i, the order of the Legendre's coefficient which corresponds to the part of the coefficient under consideration, and D, the derivative with regard to the logarithm of the ratio of the mean distances.

Before proceeding to the actual development the question of the limits within which they are convergent is considered. It is shown that when the inverse of the mutual distance between the planets is expanded in powers of σ^2

* *Amer. Jour. Math.*, Vol. III., p. 193.

(where σ is the sine of half the mutual inclination), the development is convergent provided

$$\sigma < \frac{r'-r}{2\sqrt{rr'}} \quad ,$$

r, r' being the heliocentric distances of the planets. This condition appears to be fulfilled for every pair of the eight principal planets. It may be noted that the satisfaction of this inequality gives little information concerning the stability of the solar system. The possible values which r, r' may take after the lapse of long periods of time, depend largely on the possibility of eliminating secular terms from their expressions and also on the convergence of the periodic series which are used to represent their values. The modification of the condition given above, when expansions in powers of the eccentricities are used, is deduced by considering the maximum value of the coefficient of any periodic term.

In the second paper of Vol. V., '*Inequalities of long period, and of the Second order as to the masses, in the mean longitudes oj the four inner planets,*' Professor Newcomb computes the coefficients of certain periodic inequalities, the lengths of whose periods give rise to sensible coefficients. The inequalities considered are those arising from the consideration of the mutual actions of *three* planets and the sun, the perturbations arising from the mutual action of each *pair* of planets and the sun being supposed to be known. Owing to the immense number of possibly sensible inequalities, the first step is the examination of those combinations of the mean motions which can produce an inequality of long period. In other words, if the inequality be denoted by

$$A \cos \{(in+i'n'+i''n'')t+\alpha\},$$

where n, n', n'' are the mean motions of the three planets, i, i', i'' positive or negative integers, and A, α constants, it is necessary to know for what values of i, i', i'', the expression

$$in+i'n'+i''n''$$

may be a quantity small in comparison with n, n' or n''. Tables of possible combinations, for different values of i, i' i'' satisfying this combination, are given. It is then necessary to find the corresponding coefficients.

A rough computation is first made in order to discover the numerical magnitudes of these coefficients. It is not easy to feel satisfied with the results of this computation, especially on reading the appendix, in which it is stated that two inequalities discovered by von Haerdtl and published while the memoir was passing through the press, had been overlooked. In fact, the inequalities calculated have all been previously known to have sensible coefficients and the values of the latter are given by Leverrier in nearly all cases. Quite a different impression is, however, produced with respect to the calculations undertaken for the determination of those coefficients which are selected for accurate computation. For some of these, two methods were used and there can be little doubt as to their accuracy. Moreover, Leverrier's values, which are given for the sake of comparison, agree, with one notable exception, with those obtained by Professor Newcomb, although the methods used are entirely different; the differences do not exceed two-tenths of a second, so that the coefficients may be regarded as definitive. The exception mentioned is that inequality in the motion of Mars, whose argument has a motion

$$3\,J-8\,M+4\,E,$$

J, M, E, denoting the mean motions of Jupiter, Mars and Earth, respectively. The difference amounts to one-fifth of the coefficient of the sine of the argument, the value of which, as given by Professor Newcomb, is $-40''.8$. The period of this inequality being 1800 years, it is scarcely possible, at present, to test the two values by observational means. In the motion of the Earth, the coefficient is much smaller—about $6''.4$—and the discovery of it by Hansen caused an important change in the reckoning of the Sun's mean longitude.

In the fourth part of Vol. V., Professor Newcomb discusses the *Secular Variations of the Four Inner Plants.* For this purpose he uses the method of the Variation of Arbitrary Constants applied directly to the constant part of the perturbative function, in preference to the purely numerical method of Gauss. Instead of starting with the numerical value of this constant part, the perturbative function is developed algebraically in the first instance, and then, in order to extend the results through a period of centuries, it is only necessary to insert the values of the elements for the dates chosen and to perform the few additional steps which the integration of the equations for the elements involves. If the same accuracy is required with Gauss' method, the whole of the work must be repeated for each date. The methods practically amount to finding the values arranged in powers of the time, each new date computed giving an additional term. Professor Newcomb observes that, in any case, the results are only true to the first order with respect to the masses, since the initial constant part of the perturbing function is alone used.

There is a theoretical objection to both of these methods which deserves some examination. In the construction of a planetary theory it is always desirable to have the expressions formed in such a way that only one set of numerical values shall be substituted for the arbitrary constants of the solution, *e. g.*, for the elements at a given epoch. Now the methods of procedure adopted to obtain exact values of the secular terms require that the elements at each epoch for which the variations are calculated shall be known. If we have recourse to observation to determine the elements at the different epochs the results are no longer deduced from theory alone, since we are then really using the observed values of the secular variations to obtain the values of the elements at dates differing from the standard epoch; and these secular variations are capable of being deduced from theory alone when the elements at the standard epoch are given. We may, however, compute the variations at the standard epoch, say 1850, and, by means of the values thus obtained, find approximate values of the elements at any other epoch. With these approximate values the variations at the second epoch may be computed; but it then becomes necessary to find what error has been made by taking the approximate instead of the true values of the elements at this date. A short calculation assures us that the error thus committed is of a very much higher order than the differences between the secular variations at the two dates. If, then, the observed values of the secular variations only differ from the calculated values by quantities of an order of magnitude small in comparison with the quantities sought, we may use indifferently the values of the elements as obtained by a discussion of the observations or by means of the calculated variations at the standard epoch.

In the present memoir Professor Newcomb calculates the variations at three separate dates, 1600, 1850, 2100, and for this purpose gives a table of the values of the elements at those dates. There is, however, no indication of the manner in which the elements at 1600, 2100 have been obtained (I assume that those at 1850 are deduced from the observations). From what has been said above it will be seen that this does not matter unless a large discordance exists between the different possible determinations of the elements at 1600, 2100. The only large discordance is that between the observed and calculated motion of the perihelion of Mercury which amounts to 43″ per century—about $\frac{1}{13}$ of the whole; it appears, from a short examination of the values of this element at the three dates that the observed values have been used. It might seem that this large difference would make it somewhat difficult to discover how far the observations entered into the values of the corresponding secular variations. Fortunately, the differences between the three values, and therefore the terms in t^2 are so small that either the observed or the calculated value may be used without affecting the result with an error greater than that which will arise from such exextended computations. I have dwelt at some length on this point, because one of chief objects in view is the comparison of theory with observation.

The development of the constant part of the perturbative function is made by the method referred to above. A high degree of accuracy is obtained by carrying nearly all terms to the eighth order with respect to the eccentricities and inclinations; occasionally induction is used to include terms of higher orders whenever there was a probability of the results being affected thereby. The usual canonical system of equations is then transformed by changing to variables which are analogous to those used by Hansen in his theories. Longitudes are reckoned along the orbits from departure points and the formulæ are reduced to the simplest possible forms. [In the middle of page 332, the equations for eccentricity and longitude should be

$$\frac{de}{dt} = -\frac{m'}{e\mu}\frac{a}{a'}\, n \cos\varphi \,\frac{dF}{dw} = -M_2'\frac{1}{e}\frac{dF}{dw},$$

$$e\frac{d\pi_1}{dt} = M_2'\frac{dF}{de} \qquad].$$

The results for the secular variations at the three epochs are finally obtained in forms given by the equations; the complete reduction is left until a discussion of the final values to be adopted for the masses has been made.* They

* This discussion will be found in Professor Newcomb's work, "Astronomical Constants," forming a supplement to the American Ephemeris for 1897. The final values obtained both by theory and observation for the secular variations are there given, together with an extended comparison and examination into the possible causes of the resulting differences.

are very satisfactory. The first differences between the three sets of values are in most cases constant within the limits of possible errors, showing that the terms in t^3 are negligible for a period which may be measured by several centuries.

II. Tables of the Planets.

Volume VI. contains tables for the calculation of the positions at any time of the four inner planets; of these three parts have appeared, the Tables of the Sun, of Mercury and of Venus, by Professor Newcomb; Volume VII. contains the Tables of Jupiter and Saturn, by Dr. G. W. Hill. The publication of these two volumes makes an epoch in the history of celestial mechanics which is perhaps of equal importance with that which marked the appearance of Leverrier's well-known work on the same subject. Founded on theories which have been specially worked out for the purpose, with new determinations of the constants from the additional data which have accumulated in large masses in the last thirty years, during which time observations have been made with an accuracy far greater than was previously possible, they can hardly fail to be of immense service, not only for the calculation of the annual ephemerides published by the various governments, but also for the detection of new causes of disturbance by the comparison of future observations with the tabulated places. Time alone can show to what extent they will represent the observed places, but we may confidently expect that many years will elapse before it becomes necessary to correct them to any appreciable extent.

The most important tables are naturally those of the Sun, and a little space may, perhaps, be devoted to the description of their general features. We first consider the values which have been adopted for the fundamental constants. Some of these (for example, the solar mean motion) have long been known with an accuracy which permits of little change in their values; others, especially those in which occur the perturbations (for example, the masses of certain of the planets) are yet very doubtful. The discussion of them is given in detail in Professor Newcomb's "Astronomical Constants," above referred to. It is impossible to consider here, even in a condensed form, the manner in which a balance is struck between the different values which the different classes of observations furnish. It may be mentioned that no less than 40,176 observations of the Sun alone have been reduced, and many thousands of further observations of the planets have also been included.

Of the constants whose values are most open to doubt, the solar parallax and the masses of the planets are the chief. Eight classes of observations are discussed for the determination of the former, and the values range from 8″.759 to 8″.857; the greatest weights are assigned to the values obtained from the constant of aberration.

The value adopted is:

Mean equatorial horizontal solar parallax =8″.790.

The constant of aberration is 20″.501.

The adopted masses of the planets, with the Sun as unit, are:

	Mercury,	1: 600,000
	Venus,	1: 408,000
	Earth + Moon,	1: 329,390
	Mars,	1:3,093,500
	Jupiter,	1: 1,047.35
	Saturn,	1: 3,501.6
	Uranus,	1: 22,756
	Neptune,	1: 19,540
and	Moon : Earth : : 1:81.45.	

Of these, the mass of Mercury is extremely doubtful, and it may differ from the above by one-third, or even one-half, of its whole amount; Professor Newcomb considers the assumed value to be probably a little too large. The periodic perturbations produced in the motion of the earth by this planet are, however, so small—amounting to only two or three hundredths of a second in longitude—that any admissible change in its mass will scarcely affect the tables. The above values are used in all of the tables of Vol. VI.

The chief discordance between theory and observation is in the secular motions of the perihelia, especially in that of Mercury. The possible causes of this are discussed very fully in the "Astronomical Constants." The only one which Professor Newcomb considers at all probable is that the law of attraction is not that of the inverse square, but that the index differs from 2 by a small quantity (Hall's hypothesis). He shows that with the value 2.000 000 1574 the outstanding differences in the motions of the perihelia of Mercury and Mars can be accounted for and that the effects on the other planets are within the limits of possible errors. No other sensible changes will be produced. It remains to see what effect the new law will have on the motion of the lunar perigee. On comparing Hansen's theoretical value (the only one which aspires to the required degree of precision) with the observed value, the small annual difference

(1″.56) is almost completely accounted for. But Hansen's value for the motion of the node differs from the observed value by a quantity twice as large : thus, either doubt arises as to accuracy of Hansen's calculations, or else some other cause exists affecting the motion of the node, since the above value for the index will not produce any sensible effect on this motion. As the observed motions of the lunar perigee and node are known with considerable accuracy, the question, so far as the moon is concerned, awaits a more exact theoretical determination of these motions.

The differences between theory and observation in the secular variations of the perihelia, at least in the cases of Mercury and Venus, are so well established that it has seemed advisable to take them into account in the tables. This has been done in such a manner as to make them as accurate as possible without taking away their theoretical character. Professor Newcomb considers that the value of the index instead of being 2 is

$$2 . 000\ 000\ 161\ 20.$$

The only resulting corrections are the following additions to the secular motions of the perihelia per century: Mercury, 43″.37; Venus, 16″.98; Earth, 10″.45; Mars, 5″.55.

The introduction to each set of tables is divided into parts which severally contain the values of the constants and of the periodic and secular perturbations; the formation of the arguments and the plan on which the tables are constructed; a statement of the quantities used in each table; and finally, precepts for their use. The epoch adopted is 1900.0 Greenwich mean time. The periodic perturbations are added directly to the longitude, latitude and radius vector, the arguments being so formed that, in each table of double entry, one argument is constant throughout the table. The examples given show that scarcely any multiplications and few interpolations are needed, the majority of the operations consisting of simple additions; this result is obtained in the usual manner, by adding such constants to the numbers of each table that the quantities therein are always positive; the sum of the constants added is subtracted from the constant part of each coördinate.*

* It may be useful to note here that, in the introduction to the tables of the Sun, p. 18, last line but one, the constant part to be added to $\Delta \log \nu$ should be 1600 instead of 1500. The error does not, however, affect the tables or the precepts for their use; it would only arise should the small correction to $\log \nu$, there noted, be deemed necessary.

Some statistics will best show the extent of the tables and will give a general idea of the labor involved in the calculation of an ephemeris. Those of the Sun are 38 in number, of which 6 are necessary for the formation of the arguments, 11 for the true longitude, 10 for radius vector, 4 for the latitude and 7 for the calculation of the precession, nutation, etc.; the whole occupying 136 quarto pages. Only two interpolations with second differences appear to be necessary for the longitude and radius vector and one for the latitude. The number of figures to be written in the calculation of an isolated position is about 1000; this number would, of course, be very much less, for each position, in the formation of continuous ephemeris. The results are obtained to the hundredth of a second of arc in the longitude and latitude and to 8 places in the logarithm of the radius vector.

The tables of Mercury and Venus are 22 and 27 in number, occupying 82 and 92 pages respectively, while the computation of an isolated position requires the writing of about 600 figures in each case. The results are carried to the same degree of accuracy as in the case of the Sun, with the exception of the logarithm of the solar radius vector of Mercury, which is taken to 7 places.

The tables of Jupiter and Saturn in Vol. VII. bear the signature of Dr. Hill and they are formed on the results of the theory worked out by the same writer in Vol. IV. of the series. The method there adopted was that of Hansen, with some modifications. The difficulties in obtaining a solution of the problem which these two planets present, and the somewhat unsatisfactory nature of previous theories, have caused the existing tables to present differences from the observed values which cannot be accounted for by errors of observation and which are therefore referred, in the first place, to defects in the theory or in the values of the constants used. In the "New Theory of Jupiter and Saturn," Dr. Hill has adopted the numerical method with values of the elements which are approximate, but which may be supposed to differ from the true values by certain appreciable though small quantities.

In order to obtain more correct values of the elements, Dr. Hill has formed provisional tables from his theory and has compared them with abont 19,000 observations of the two planets, extending from 1750 to 1865; a little more than half of these being in right ascension and the rest in declination. Normal equations having been formed, the corrections to the values of the elements are obtained and the necessary changes in the theory follow. The main results of this investigation are given in the introductions to the tables. The mass of Jupiter, which results from the perturbations of Saturn by this planet is $1 : 1047.378 \pm .121$, and the mass of Saturn, from the perturbations of Jupiter is $1 : 3502.20 \pm .53$. These are very close to the values adopted by Professor Newcomb (see above), and the latter are used in the tables.

The tables are not uniform with those of Volume VI. The epoch is taken to be 1850.0, G. M. T., while the masses of Uranus and Neptune differ slightly. Newcomb's correction to the motions of the perihelia is insensible in the cases of these planets. Further, the form of the tables, owing to the use of Hansen's method, is quite different. The latter are 68 in number for Jupiter and 71 for Saturn, but the space occupied by them, and the number of figures to be written for the calculation of an isolated position, are about the same. In order to ensure the accuracy required and at the same time not to increase the size of the tables by the use of an extra decimal place, Dr. Hill adopts a device of Leverrier. The periodic inequalities of the mean anomaly and of the logarithm of the radius vector have been multiplied by 3; the sums of the results when these are taken out are to be divided by the same number. The difficulty which it is required to avoid is brought about by the use of the decimal scale of notation, which gives a certain degree of accuracy for a given number of places of decimals, or an accuracy ten times as great by the use of another place. The accuracy required is one midway between these two, and this is obtained without adding another decimal place by the artifice just mentioned. Full details are given as to the construction of the tables, with specific directions for their use.

III. *Action of the Planets on the Moon.* Vol. V. Pt. III.

The problem involved in the investigation of the effects of planetary action on the motion of the Moon is generally regarded as one of the most difficult in the whole range of celestial mechanics. Besides the usual difficulty which always arises in the investigation of any part of the lunar motion, there is added another which occurs in the planetary theory, namely, the slow convergence of the series in the cases of those planets which are not far from the earth and the consequent effect on the coefficients whenever the periodic terms have long periods. Further, it is usual and frequently necessary in the lunar theory to avoid all terms in which the time occurs outside the signs, sine and cosine and the number of combinations which the different arguments give rise to becomes very great. Until the last few years the problem has been attacked chiefly from its practical side, and few theoretical results have been obtained. A first glance at the reading of Professor Newcomb's memoir seems to indicate a wide field for investigations into the theory; in any case the practical results, notwithstanding the labors of Hill, Radau, Newcomb and of the earlier investigators, leave much to be desired. The problem is in reality a case of the disturbed motion of three bodies, rather than a further extension of the problem of two bodies, and it is from this point of view that Professor Newcomb attacks it in the memoir under consideration here.

The paper opens with a more rigorous investigation of the results of the lunar theory when the disturbing action of the Sun is alone considered. The general equations of the motions of the bodies are first found and reduced to six of the second order, rectangular coördinates being used. Three of these equations refer to the motion of the Moon with respect to the Earth, and the other three to that of the Sun with respect to the centre of the mass of the Earth and Moon, the bodies being considered throughout as spheres. The method of solving these equations is well known. The Sun is first supposed to move in an elliptic orbit, and with this approximation a first approximation to the motion of the Moon is found; the result is contained in Delaunay's 'Théorie de la Lune.' Returning with these values of the lunar coördinates to the equations for the solar motion, more approximate values of the solar coördinates are found, and thence, by considering again the lunar equations, a second approximation to the lunar coördinates. Owing to the minuteness of the new terms furnished by the second approximations, we are enabled to stop at this point and to consider the problem of three bodies, so far as it refers to the Earth, Moon and Sun, as solved.

Professor Newcomb now introduces a fourth body and considers the effect of its disturbing action on the motions of the other three bodies. It is at this point that he makes a departure from the plan adopted by previous investigators. The latter have usually considered these perturbing effects separately, that is to say, they have found separately the perturbations produced on the Moon, those produced on the apparent motion of the Sun and, as a further step, the effects of the resulting perturbations of the Sun on the motion of the Moon. The method adopted to obtain them is that of making, successively, the six elements of each motion vary in such a manner that the new terms may be included. Professor Newcomb considers the *twelve* elements of the Moon and Sun as variable and he thus includes at once all the three sets of disturbances; his method is therefore the disturbed motion of *three* bodies, and the new results which he obtains are mainly due to this plan of considering the perturbations.

The treatment is, at the outset, a general application of Lagrange's method for the variation of arbitrary constants. Representing the twelve arbitraries by $a_1, a_2, \cdots a_{12}$, the chief labor involved is the calculation of the Lagrangian coefficients (a_i, a_j)—the coefficients of the derivatives of $a_1, a_2 \cdots$ with respect to the time. The fact that these coefficients do not contain the time explicitly is the main factor in the determination of their values.

The six rectangular coördinates of the Moon referred to the Earth, and of the Sun referred to the centre of mass, of the Earth and Moon are known to be expressible in the forms,

$$x \text{ or } x' = \Sigma k \cos N,$$
$$y \text{ or } y' = \Sigma k \sin N,$$
$$z \text{ or } z' = \Sigma k \sin N',$$

where $$N \text{ or } N' = i_1 \lambda_1 + i_2 \lambda_2 + \cdots + i_6 \lambda_6;$$

in which, for N,

$$i_1 + i_2 + \cdots + i_6 = 0,$$

and for N'

$$i_1 + i_2 + \cdots + i_6 = 1.$$

Also, $$\lambda_i = l_i + b_i t.$$

In these expressions $l_1, l_2 \cdots l_6$ are six of the arbitrary constants of the solution, the coefficients k of the periodic terms and b_i of the time in the angles, being functions of six other arbitrary constants. The time thus enters into the expressions for the coördinates only through the angles λ_i in the manner just stated.

It is now possible to undertake the calculation of the Lagrangian coefficients (a_i, a_j). Six of the arbitraries are considered to be the l_i and the other six, $a, e, \gamma, a', e', \gamma'$, of which the k and b_i are functions. These coefficients are divided into parts according to the way in which they are derived, certain terms which are known to be quite insensible being neglected. The result obtained is as follows: Half of the whole number of Lagrangian coefficients are zero, and the rest can be expressed as the partial derivatives of six functions $c_1 c_2 \cdots c_6$ with respect to $a, e, \gamma, a', e', \gamma'$, the six functions containing only the latter six arbitraries. At this point, it is a simple step to obtain canonical equations for the variations of the elements. They are, in fact,

$$\frac{d c_i}{d t} = \frac{\partial R}{\partial l_i}, \quad \frac{d l_i}{d t} = -\frac{\partial R}{\partial c_i}, \quad (i = 1, 2, \cdots 6).$$

A few remarks on this result may, perhaps, be useful. It is known with respect to canonical systems of equations, that when half the arbitrary constants have been chosen, the form in which the rest of the arbitraries enter into the integrals is, in general, definite. It is natural to choose the l_i as six of the arbitrary constants, and the problem is therefore reduced to an investigation of the manner in which the other six arbitraries c_i enter into the expressions for the coördinates in order that the system may be canonical. The method used by Professor Newcomb to solve this problem, that is, to find the relations of the c_i to the coefficients k, b_i is powerful, if somewhat long, and one cannot but admire the manner in which the symbols are arranged and dealt with in order to bring about the desired result. Nevertheless it would seem possible to obtain it more easily by the use of the partial differential equation satisfied by the principal function. Jacobi has shown that if a solution of this partial differential equation, involving the coördinates q_i, the time, and half the required number of arbitrary constants can be obtained, the integrals of the problem are given by

$$p_i = \frac{\partial S}{\partial q_i}, \quad l_i = \frac{\partial S}{\partial c_i},$$

where S denotes the principal function, and p_i, q_i satisfy Hamilton's equations. The property which this method possesses is contained in the fact that the c_i and the l_i form a canonical system. In the present case, therefore, the problem reduces to the discovery of any solution of the equation for S, involving the coördinates, the time and the six arbitraries c_i. The difficulty involved in the discovery of such a solution generally makes it impossible to use the Jacobian method as a first means of investigation, and it is, in consequence, chiefly resorted to as a brief method of proving known results. No solution of the equation appears to be readily available for the proof of Professor Newcomb's results.*

The kinetic energy of the system of three bodies is given by

$$2T = \mu_1 (x^2 + y^2 + z^2) + \mu_2 (x'^2 + y'^2 + z'^2)$$

where μ_1, μ_2 depend on the masses of the three bodies. The force function is Ω where

$$\Omega = \frac{m_1 m_2}{\rho_{12}} + \frac{m_1 m_3}{\rho_{13}} + \frac{m_2 m_3}{\rho_{23}}$$

m_1, m_2, m_3 being the three masses. The function c_i, which is first to be considered as an arbitrary constant, is obtained by summing the products

$$k^2 (i_1 b_1 + i_2 b_2 + \cdots + i_6 b_6) i_i$$

for every periodic term in each of the six coördinates, multiplying those portions which arise from x, y, z by μ_1, and those portions which arise from x', y', z' by μ_2. This result, remarkable as it is from a theoretical point of view, is not of less interest practically, since it enables us to use canonical constants after solving the problem of three bodies and obtaining the values of the rectangular coördinates in terms of the constants of the solution.

The further result then follows that

$$b_i = -\frac{\partial T_o}{\partial c_i}$$

where T_0 is the non-periodic part of T when expressed in terms of the time as a sum of periodic terms. In the 'General Integrals of Planetary Motion,'† Professor Newcomb, has further shown that

$$T_0 = -C,$$

where

$$T - \Omega = C.$$

Hence, when C has been expressed in terms of the c_i, the secular motions of the six angles are obtained by taking the partial derivatives of the constant of energy with respect to the c_i. This cannot, of course, be used to find them, since

* Since the above was written I have succeeded in proving the results obtained by Professor Newcomb in the manner indicated, and have also obtained some new formulæ which include the theorems of Adams mentioned below. These will be published shortly.

† Smithsonian Contributions to Knowledge, Vol. XXI.

the solution of the problem must be first obtained in order to find T_0 or C. A comparison of these results with those obtained by Adams concerning the relations of the secular motions of the lunar perigee and node, with the constant part of the lunar parallax,* is suggestive.

The next step is to obtain the values of the constants c_i from the known expressions for the coördinates of the Sun and Moon. The angles λ_i are defined to be the mean positions of the two bodies and of their perigees and nodes from some fixed line. When the relation of the theory to that of Delaunay is under consideration, this definition gives a simple relation between c_1, c_2, c_3 (which refer to the Moon) and the values of Delaunay's L, G, H after the last operation. Neglecting insensible terms, we obtain from the fact that a transformation of the kind known as "tangential" retains the canonical form of the equations, the relations

$$k_\epsilon = \mu_2 L, \quad k_\pi = \mu_2(G-L), \quad k_\theta = \mu_2(H-G),$$

where k_ϵ, k_π, k_θ, replacing c_1, c_2, c_3, denote the constants reciprocal to ε_0, π_0, θ_0 (the angles which give the mean positions of the Moon, its perigee and its node at time $t = 0$). The other parts of k_ϵ, k_π, k_θ, arising from the perturbations of the solar orbit on the lunar orbit, being negligible, k_ϵ, k_π, k_θ, might have been obtained from Delaunay's final values for L, G, H. Professor Newcomb, however, finds them directly by transforming Delaunay's final expressions for the longitude, latitude and parallax to rectangular coördinates and then computing them by the rules stated above. The results are expressed in terms of the final constants a, e, γ, a', e' used by Delaunay. He thus obtains an important verification of the accuracy of Delaunay's expressions. The values of L, G, H, given on pages 235, 236 of Vol. II. of Delaunay's work, are obtained in terms of his final constants by means of the formulæ given on page 800 of the same volume, and the results obtained by the two methods agree as far as the order to which they are carried. The labor of forming k_ϵ, k_π, k_θ is great and it occupies a large part of Professor Newcomb's memoir. The chief parts of $k_{\epsilon'}$, $k_{\pi'}$, $k_{\theta'}$ are those due to the elliptic motion; the other portions due to the action of the Moon are very small, though not insensible.

The equations for the variations of the elements having been obtained, we arrive at formulæ which will enable us

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*M. N. R. A. S., Vol. XXXVIII., pp. 460–472. See also a paper by the writer in the Amer. Jour. Math., Vol. XVII., pp. 318–358.

to calculate the effect of the action of a planet on these elements. The new part of the disturbing function is found in the usual way. The planetary action is usually divided into two parts, the *direct* action, arising from the attraction of the planet on the Moon, and the *indirect* action, arising from the attraction of the planet on the Earth and transmitted through the latter to the Moon. Professor Newcomb first considers the latter. Suppose the variations of the elements of the solar orbit have been obtained and enquiry be made as to the effect of these variations on the elements of the lunar orbit. For the latter we have, in general,

$$\frac{dk_\epsilon}{dt}=\frac{\partial R}{\partial \varepsilon}, \quad \frac{d\varepsilon}{dt}=-\frac{\partial R}{\partial k_\epsilon}$$

with two similar pairs of equations. The terms arising from R are those due to the direct actions of the planets. For the indirect actions we therefore have

$$\frac{dk_\epsilon}{dt}=0, \quad \frac{d\varepsilon}{dt}=0, \text{ etc.}$$

Now k_ϵ, k_π, k_θ are found in terms of a, e, γ, a', e', (γ' being neglected. If, then, we are considering the indirect action, and if, further, we suppose the perturbations of a', e' known, the perturbations of a, e, γ are obtained by solving the three equations

$$\frac{\partial k_\epsilon}{\partial a}\frac{da}{dt}+\frac{\partial k_\epsilon}{\partial e}\frac{de}{dt}+\frac{\partial k_\epsilon}{\partial \gamma}\frac{d\gamma}{dt}+\frac{\partial k_\epsilon}{\partial a'}\frac{da'}{dt}+\frac{\partial k_\epsilon}{\partial e'}\frac{de'}{dt}=0,$$

$$\frac{\partial k_\pi}{\partial a}\frac{da}{dt}+\cdots\cdots=0, \quad \frac{\partial k_\theta}{\partial a}\frac{da}{dt}+\cdots\cdots=0.$$

If quantities of the first order, with respect to the perturbations of a', e', are alone considered, we finally arrive at the result that the indirect actions of the planets are obtained by solving the equations

$$\delta k_\epsilon=0, \quad \delta k_\pi=0, \quad \delta k_\theta=0,$$

or, in Delaunay's notation,

$$\delta L=0, \quad \delta G=0, \quad \delta H=0.$$

In other words, *if the coefficients in the time in the arguments* *

* These must be expressed in the usual forms. Thus for the mean motion we put $\int n\, dt$ instead of nt.

and of the periodic terms in Delaunay's results were all expressed in terms of L, G, H, a', e', the perturbations due to the indirect actions of the planets would be obtained by merely inserting the variable instead of the constant values of a', e'.

The application of this remarkable result to the determination of the secular accelerations is immediate, since these inequalities depend on the variability of the solar eccentricity. It is only necessary to express Delaunay's final values of L, G, H in terms of his final constants, n or a, e, γ; the variations of a, e, γ are then obtained by solving the equations

$$\frac{\partial L}{\partial a}\delta a + \frac{\partial L}{\partial e}\delta e + \frac{\partial L}{\partial \gamma}\delta \gamma + \frac{\partial L}{\partial e'}\delta e' = 0\,,$$

$$\frac{\partial G}{\partial a}\delta a + \cdots = 0\,, \quad \frac{\partial H}{\partial e}\delta a + \cdots = 0,$$

with the given value of $\delta e'$. The secular accelerations are obtained by inserting these values for δa, δe, $\delta \gamma$, $\delta e'$ in $\int dt\delta n$, $\int dt\,\delta\pi_1$, $\int dt\,\delta\theta_1$ where n, π_1, θ_1 are the mean motions of the Moon and of its perigee and node. Professor Newcomb has verified this result in the case of the mean motion of the Moon. The application of the theorem to those periodic terms which are due to the indirect action of the planets is made in a similar way.

Professor Newcomb remarks that "this curious theorem may embody some principle applicable to the disturbed motion of three bodies which has not yet been fully mastered." It seems probable, from the way in which the result has been obtained, that it is a direct consequence of the use of canonical equations and of the form in which the time appears in the result. Canonical equations are the only ones at present known which will give symmetrical forms for researches into celestial mechanics, and the possibility of obtaining forms which can be made practically useful depends, to a large extent, on the fact that we have a system of constants (half the whole number required), ready to hand in the constant parts of the angles which occur under the signs, sine and cosine. The additional properties, first, that the force function does not contain the time explicitly, and secondly, that the coefficients of the time in the arguments can be expressed as the derivatives of the constant of energy, may indicate lines on which further investigations into the problem of three bodies might be carried, especial attention being directed to the properties of canonical systems and especially of those of Delaunay and Newcomb.

Passing on to the consideration of the terms due to the direct action of the planets, Professor Newcomb obtains forms for the perturbations of the elements, with numerical coefficients similar to those obtained by Hill and Radau. He then considers the development of the perturbing function for planetary action after Hansen's method, and application is made to the cases of the principal inequalities arising from the various planets. Examination is also made into the possible cases where inequalities may become sensible owing to the presence of large divisors, the whole subject being dealt with in detail. There are, notwithstanding the researches made of late years into this subject, some outstanding inequalities of long period in the motion of the Moon which have not yet been satisfactorily accounted for by theory. I close with the remarks of Professor Newcomb on this subject. At the end of the Preface to his Memoir he says: "There are three possible sources to which may be attributed the apparent inequalities in question:

(α) Defects in the gravitational theory.

(β) Inequalities in the Earth's time of rotation.

(γ) Action on the Moon of other forces than the gravitation of known bodies.

The second of these causes seems to be not at all improbable, in view of recent discoveries respecting variations of latitude, which can only be accounted for by minute changes in the Earth's axis of rotation. But the independent tests of this cause offered by the observed Transits of Mercury do not show that it suffices to explain the observed inequalities in the motion of the Moon. Two other tests remain to be applied: The motion of the first satellite of Jupiter and observations of Mercury, Venus and Mars. So far as the evidence on these points has been brought out, the conclusion indicated is that inequalities in the Earth's rotation are real, but not large enough to produce the observed effects on the Moon's mean longitude. The subject is therefore still involved in doubt, and the first step toward the removal of this doubt is such a determination of the inequalities in the mean motion produced by the action of the planets as shall place the subject on an undoubted basis."

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HAVERFORD COLLEGE,
May 1, 1896.