ON A GENERAL FORMULA FOR THE EXPANSION OF FUNCTIONS IN SERIES.*

BY PROF. W. H. ECHOLS.

1. If O be the symbol which represents any operation performed on a function, and O^r the repetition of that operation r times, then the formula referred to above is

$$\left|\begin{array}{ccccc} fx & f_1x & \ldots & f_nx & f_{n+1}x \\ fy_1 & f_1y_1 & \ldots & f_ny_1 & f_{n+1}y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fy_p & f_1y_p & \ldots & f_ny_p & f_{n+1}y_p \\ Ofx_1 & Of_1x_1 & \ldots & Of_nx_1 & Of_{n+1}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ O^qfx_q & O^qf_1x_q & \ldots & O^qf_nx_q & O^qf_{n+1}x_q \\ \Phi(u) & 0 & \ldots & 0 & 1 \end{array}\right| \qquad (1)$$

in which all elements of the last row except the first and last are zero. The symbol O^rfx_t means that after the rth operation on fx, the argument is changed into x_t. $\Phi(u)$ represents, in general, some function of $x, y_1, \ldots, y_p, x_1, \ldots, x_q$, involving also the form of the functions in the determinant.

If now the operation O be such that the Φ function may be so determined that the above determinant vanishes, we have, regarding x as the variable, the formulæ

$$fx = A_1f_1x + \ldots + A_{n+1}f_{n+1}x,$$
$$fx = B_1fy_1 + \ldots + B_pfy_p$$
$$+ C_1Ofx_1 + \ldots + C_qO^qfx_q + D\Phi(u).$$

The first of these may be regarded as an expansion of fx according to the functions $f_1x, \ldots f_{n+1}x$, whose coefficients are independent of the argument x, save in so far as Φ is a function of x. The second, in turn, may be regarded as an expansion of fx according to the form fy_r and the successive operatives of fx, whose coefficients are independent of the form of the function fx; the residual term being $D\Phi(u)$, wherein D does not depend on the form of the function fx.

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* Read before the New York Mathematical Society, January 7, 1893. This paper is intended to be a brief exposition of the general theorem which is the basis of a series of papers entitled 'On Certain Determinant Forms and their Applications,'' now in course of publication in the *Annals of Mathematics.*

We proceed to examine particular cases of (1) wherein the determinant may be either that of Interpolation, Differentiation, or Differences.

INTERPOLATION.

2. In (1), if $q = 0$, then each row, except the last, is obtained by specifying the variable in the first row, and we have, for the general formula of interpolation,

$$\begin{vmatrix} fx, & f_1x & \ldots & f_nx, & f_{n+1}x \\ fx_1, & f_1x_1 & \ldots & f_nx_1, & f_{n+1}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fx_n, & f_1x_n & \ldots & f_nx_n, & f_{n+1}x_n \\ \Phi(u), & 0 & \ldots & 0, & 1 \end{vmatrix} = 0, \qquad (2)$$

wherein

$$\Phi(u) = \begin{vmatrix} f^nu, & f_1^nu & \ldots & f_n^nu \\ fx_1, & f_1x_1 & \ldots & f_nx_1 \\ \cdot & \cdot & \cdot & \cdot \\ fx_n, & f_1x_n & \ldots & f_nx_n \end{vmatrix} \div \begin{vmatrix} f_1^nu & \ldots & f_{n+1}^nu \\ f_1x_1 & \ldots & f_{n+1}x_1 \\ \cdot & \cdot & \cdot \\ f_1x_n & \ldots & f_{n+1}x_n \end{vmatrix}$$

in which f_r^nu means that f_rx is to be differentiated n times with respect to x and in the result x changed into u, which is some unknown quantity lying in value between the greatest and least values of the quantities $x, x_1, \ldots, x_n$.

The proof follows:

Let M be the minor of $\Phi(u)$, and N be the minor of the element 1 in (2), and put

$$N = (-1)^n MR,$$

R being some unknown function of x.

Let us now assign to x some arbitrary constant value x_0, so that this equality becomes

$$N_0 = (-1)^n M_0R_0,$$

which is independent of x.

Consider the function

$$F = N + (-1)^{n+1}MR_0.$$

This function vanishes when $x = x_0$ and also when x is equal to any one of the n quantities $x_1, \ldots, x_n$. By Rolle's theorem, therefore, the first derivative of F must vanish for n values of x such as $u_1, \ldots, u_n$, which lie respectively between the values $x_0x_1, x_1x_2, \ldots, x_{n+1}x_n$. In like manner the second derivative of F vanishes for $n-1$ values of x which lie respectively between the values u_1u_2, u_2u_3, . . . $u_{n-1}u_u$; and so on, until finally the nth derivative of F must vanish for some value, u, of x, which lies between the greatest and the least of the quantities $x_0, x_1, \ldots, x_n$, and we have

$$F_u^n = N_u^n + (-1)^{n+1} M_u^n R_0 = 0.$$

Since x_0 is arbitrary, we may drop the subscript, and write

$$R = (-1)^n N_u^n / M_u^n,$$
$$= \Phi(u).$$

Whence

$$N + (-1)^{n+1}\Phi(u) M = 0,$$

which demonstrates (2).

As we shall require, in the sequel, the result of the following, we proceed to give a particular illustration:

Let $f_r x = x^{r-1}$, then

$$\Phi(u) = \frac{f^n(u)}{n!}.$$

and we have

$$\begin{vmatrix} fx, & 1, & x & \ldots & x^{n-1}, & x^n \\ fa_1, & 1, & a_1 & \ldots & a_1^{n-1}, & a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & a_n & \ldots & a_n^{n-1}, & a_n^n \\ \frac{f^n(u)}{n!}, & 0, & 0 & \ldots & 0, & 1 \end{vmatrix} = 0. \qquad (3)$$

Expanding this with respect to the first column, we obtain

$$fx = A_1 fa_1 + A_2 fa_2 + \ldots + A_n fa_n + A_{n+1}\frac{f^n(u)}{n!},$$

wherein

$$A_r = (-1)^{r+1}\frac{\zeta^{\frac{1}{2}}(x, a_1 \ldots a_{r-1}, a_{r+1} \ldots a_n)}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)},$$

$$= \frac{(x-a_1)\ldots(x-a_{r-1})(x-a_{r+1})\ldots(x-a_n)}{(a_r-a_1)\ldots(a_r-a_{r-1})(a_r-a_{r+1})\ldots(a_r-a_n)}.$$

This is Lagrange's interpolation formula.

Let the values of the argument be equidistant with increment h, x being the greatest so that $x - a_r = rh$, then we have (E being the symbol of enlargement)

$$(-)_n h^n f^n(u) = fx - C_{n_1} E' fx + \ldots + (-)^{r+1} C_{nr} E^r fx + \ldots + (-)^n E^n fx.$$

C_{nr} representing the binomial coefficient, and $E^r fx = f(x - rh)$. The member on the right of this equality is the well-known expression for the nth difference of fx, so we have

$$\Delta^n fx = h^n f^n(u), \tag{4}$$

wherein u lies between x and $x - nh$. If $n = 1$, then

$$f(x + h) - fx = hf'(u),$$

Lagrange's well-known form of Rolle's theorem. We may therefore consider (4) to be a generalization of this formula.

DIFFERENTIATION.

3. If in (1) the operation O be identical with the operation of Differentiation, we have for the corresponding general formula

$$\begin{vmatrix} fx, & f_1 x & \ldots & f_n x, & f_{n+1} x \\ fy_1, & f_1 y_1 & \ldots & f_n y_1, & f_{n+1} y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fy_p, & f_1 y_p & \ldots & f_n y_p, & f_{n+1} y_p \\ f'x_1, & f_1' x_1 & \ldots & f_n' x_1, & f'_{n+1} x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^q x_q, & f_1^q x_q & \ldots & f_n^q x_q, & f^q_{n+1} x_q \\ \Phi(u), & 0 & \ldots & 0, & 1 \end{vmatrix} = 0. \tag{5}$$

In which, as before, u is an unknown value of x lying between the greatest and least of the quantities $x, y_1, \ldots, y_p, x_1, \ldots, x_q$. The bottom element of each column except the first and last is zero, and*

* In point of fact we should in the general form (5) write

$$\Phi(u) = \left(\frac{d}{dx}\right)^{n=p+q}_{x=u} \begin{vmatrix} fx & \ldots & f_n x \\ fy_1 & \ldots & f_n y_1 \\ \cdot & \cdot & \cdot \\ fy_p & \ldots & f_n y_p \\ f'x_1 & \ldots & f'_n x_1 \\ \cdot & \cdot & \cdot \\ f^q x_q & \ldots & f_n^q x_q \end{vmatrix} \div \left(\frac{d}{dx}\right)^{n=p+q}_{x=u} \begin{vmatrix} f_1 x & \ldots & f_{n+1} x \\ f_1 y_1 & \ldots & f_{n+1} y_1 \\ \cdot & \cdot & \cdot \\ f_1 y_p & \ldots & f_{n+1} y_p \\ f_1' x_1 & \ldots & f'_{n+1} x_1 \\ \cdot & \cdot & \cdot \\ f_1^q x_q & \ldots & f^q_{n+1} x_q \end{vmatrix},$$

because F vanishes $p+1$ times for $x = x_0, y_1, \ldots, y_p$, therefore its

$$\Phi(u)=\begin{vmatrix} f^{q+1}u & \cdots & f_n^{q+1}u \\ fy_1 & \cdots & f_ny_1 \\ \cdot & \cdots & \cdot \\ fy_p & \cdots & f_ny_p \\ f'x_1 & \cdots & f_n'x_1 \\ \cdot & \cdots & \cdot \\ f^qx_q & \cdots & f_n^qx_q \end{vmatrix} \div \begin{vmatrix} f_1^{q+1}u & \cdots & f_{n+1}^{q+1}u \\ f_1y_1 & \cdots & f_{n+1}y_1 \\ \cdot & \cdots & \cdot \\ f_1y_p & \cdots & f_{n+1}y_p \\ f_1'x_1 & \cdots & f'_{n+1}x_1 \\ \cdot & \cdots & \cdot \\ f_1^qx_q & \cdots & f^q_{n+1}x_q \end{vmatrix}$$

The proof follows:

Let M and N be the minors of $\Phi(u)$ and 1, respectively, in (5), and put

$$N=(-1)^nMR,$$

R being some unknown function of x. Assigning to x some arbitrary constant value x_0, we have

$$N_0=(-1)^nM_0R_0.$$

Consider the function

$$F=N+(-1)^{n+1}MR_0.$$

F vanishes when x takes any one of the values $x_0, y_1, \ldots, y_p$. Therefore, by the above, its derivative must vanish for some value of x, say u_0, which lies between the greatest and least of these values. This derivative vanishing also for $x=x_1$, then must the second derivative vanish for some value u_1, which lies between u_0 and x_1; which, in turn, vanishes again for $x=x_2$. Continuing thus, we find that the $(q+1)$th derivative of F vanishes for some value, u, of x lying between the limits prescribed above. Therefore

$$F_u^{q+1}=N_u^{q+1}+(-1)^{n+1}M_u^{q+1}R_0=0.$$

x_0 being arbitrary, we may strike off the subscript and put

$$R=(-1)^nN_u^{q+1}/M_u^{q+1}$$
$$=\Phi(u).$$

Whence

$$N+(-1)^{n+1}\Phi(u)M=0,$$

which is (5).

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first derivative vanishes p times between these values and also once more when $x=x_1$, and so on, until we find its qth derivative vanishing $p+1$ times among the values $x_0, y_1, \ldots, y_p, x_1, \ldots, x_q$, so that the $(q+p)$th or nth derivative must vanish once among them. The same thing would apply to the general formula for differences, etc.

The most interesting case of this general formula is when $p = 1$. It may then be written

$$\begin{vmatrix} fx, & 1, & \phi_1 x, \ldots, & \phi_n x, & \phi_{n+1} x \\ fy, & 1, & \phi_1 y, \ldots, & \phi_n y, & \phi_{n+1} y \\ f'a_1, & 0, & \phi_1' a_1, \ldots, & \phi_n' a_1, & \phi'_{n+1} a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n, & 0, & \phi_1^n a_n, \ldots, & \phi_n^n a_n, & \phi^n_{n+1} a_n \\ \Phi(u), & 0, & 0, \ldots, & 0, & 1 \end{vmatrix} = 0. \quad (6)$$

I have for want of a better name called the general form a composite, and the minor of fx is designated by the term *body-determinant*, or simply the *body* of the composite, in as much as it is got by striking out the outside rows and columns of the composite.

There are two classes of ϕ functions in (6) which require classification. The first class includes all of those functions which yield a body such that all elements on one side of its diagonal vanish, either through the operation of differentiation alone or through proper selection of the arbitrary constants. This class may be subdivided according as the elements above or below the diagonal vanish.* The second class includes those cases in which the elements on neither side of the body diagonal all vanish, the most interesting case of this class being that in which the body is a difference-product.

The first class yields readily all the well-known series, such as those of Taylor, Maclaurin, Bernouilli, Lagrange, Laplace, Abel, and a large number of other general series. The second class yields Fourier's theorem, and important general series in sines, cosines, Bessel's functions and logarithmic forms. A large number of these forms I have deduced in detail in the *Annals of Mathematics*, VI., 5 ; VII., 1, etc., with the object in view of illustrating the application of the composite to the deduction of special forms.

DIFFERENCES.

4. After demonstrating the general formula for interpolation we took notice of a special case for the purpose of deducing (4), the generalization of Lagrange's form of Rolle's theorem, because that theorem will now be needed for the establishment of the corresponding general formula for Finite Differences, which is, in the form corresponding to (6), as follows :

* The first division of this class is Wronski's expansion.

$$\begin{vmatrix} fx, & 1, & \phi_1 x \ldots & \phi_n x, & \phi_{n+1} x \\ fy, & 1, & \phi_1 y \ldots & \phi_n y, & \phi_{n+1} y \\ \Delta f x_1, & 0, & \Delta \phi_1 x_1 \ldots & \Delta \phi_n x_1, & \Delta \phi_{n+1} x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^n f x_n, & 0, & \Delta^n \phi_1 x_n \ldots & \Delta^n \phi_n x_n, & \Delta^n \phi_{n+1} x_n \\ \Phi(u), & 0, & 0 \ldots & 0, & 1 \end{vmatrix} = 0. \quad (7)$$

The proof follows :

Using M, N, and R in the same sense as before, we consider the function

$$F = N + (-1)^{n+1} M R_0 .$$

$F = 0$ when $x = x_0$ and also when $x = y$; therefore its first derivative F' vanishes for some value of x which lies between x_0 and y, say u_0. Now when $x = x_1$, then $\Delta F = 0$; hence, if the scale of difference be h, in virtue of (4)

$$\Delta F = hF'(u)$$

(u lying between x and $x + h$), we have $F' = 0$ for some value of x between x_1 and $x_1 + h$, say $x_1 + h_1$. Since $F' = 0$ for u_0 and $x_1 + h_1$, then must $F'' = 0$ for some value of x, say u_1, between u_0 and $x_1 + h_1$.

Again, since by (5) we have

$$\Delta^2 Fx = h^2 F''(u)$$

(u between x and $x + 2h$), and since $\Delta^2 Fx = 0$ when $x = x_2$, then must $F'' = 0$ for some value of x between x_2 and $x_2 + 2h$, say $x_2 + h_2$. F'' vanishing for $x = u_1$ and $x = x_2 + h_2$, then must $F''' = 0$ for some value of x, say u_2, which lies between these values.

Reasoning in the same way, we proceed until finally we show that the $(n + 1)$th derivative of F must vanish for some value u, of x, which lies between the greatest and the least of the quantities x_0, y, $x_1 + h$, . . ., $x_n + nh$, so that we have

$$F_u^{n+1} = N_u^{n+1} + (-1)^{n+1} M_u^{n+1} R. = 0.$$

Dropping the suffix as before, we obtain

$$N + (-1)^{n+1} \Phi(u) M = 0,$$

which is (7).

Interesting forms of (7) are of course the general expansions in factorials, a number of which I have deduced, including as special cases the generalized forms of Taylor's and Maclaurin's series.

5. All expressions in the form of series which are deduced from the general composite are to be considered as having a finite number of terms and a terminal term R. These formulæ are not to be imagined as extending to infinity until it has been demonstrated that R becomes evanescent when n is infinite, and the coefficients in the series at the same time take on finite form. In general, it will be required of the functions in the composite, that they be finite, continuous, and single-valued between the values of the argument indicated, as also their successive operations. Under these circumstances when the member on the right converges to fx, on the left, as a limit when n becomes infinitely large, the result may be relied upon as arithmetically intelligible and true, the residual term R furnishing the evidence as to what values of the variable may be used in the equality. When we have determined the forms of the coefficients in the series, we have definitely determined the true analytical forms which these series must have if the expansion be possible, that is to say, we may regard the *qualitative* analysis as having been effected for these formulæ. The *quantitative* analysis remains yet to be done, that is the investigation of R, which determines the existence of the converging infinite series and the limits between which the variable can lie. The terminal term R is a function of an absolutely unknowable value of x, which can only be eliminated in the limit by showing that R vanishes when n is infinite.

The *rationale* illustrating the application of the composite to the expression of functions in infinite series may be presented thus :

Let there be two functions fx and

$$\sum_{0}^{n-1} A_r\phi_r x = A_0 + A_1\phi_1 x + \ldots + A_{n-1}\phi_{n-1}x.$$

Let the difference between these two functions be R, so that

$$fx = A_0 + A_1\phi_1 x + \ldots + A_{n-1}\phi_{n-1}x + R. \tag{8}$$

Let $a_1, \ldots, a_n$ be certain arbitrary values of the variable x, and let us have

$$\left.\begin{array}{l} fa_1 = A_0 + A_1\phi_1 a_1 + \ldots + A_{n-1}\phi_{n-1}a_1 + R_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ fa_n = A_0 + A_1\phi_1 a_n + \ldots + A_{n-1}\phi_{n-1}a_n + R_n \end{array}\right\}. \tag{9}$$

In these n relations (9) there are n undetermined arbitrary quantities $A_0, A_1, \ldots, A_{n-1}$. Let us determine these so that we shall have $R_1 = R_2 = \ldots = R_n = 0$. Thus, the value of A_r which satisfies this condition is,

$$A_r = (-1)^r \frac{\begin{vmatrix} fa_1, & 1, & \phi_1 a_1 & \ldots & \phi_{r-1} a_1, & \phi_{r+1} a_1 & \ldots & \phi_{n-1} a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & \phi_1 a_n & \ldots & \phi_{r-1} a_n, & \phi_{r+1} a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}}{\begin{vmatrix} 1, & \phi_1 a_1 & \ldots & \phi_{n-} a_1 \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \phi_1 a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}}.$$

Consider the A coefficients to have these values. Taking now the $n+1$ relations (1) and (2), we have for the value of R,

$$\frac{\begin{vmatrix} fx, & 1, & \phi_1 x & \ldots & \phi_{n-1} x \\ fa_1, & 1, & \phi_1 a_1 & \ldots & \phi_{n-1} a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & \phi_1 a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}}{\begin{vmatrix} 1, & \phi_1 a_1 & \ldots & \phi_{n-1} a_1 \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \phi_1 a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}} = R.$$

We observe that the expansion of the determinant in the numerator of this ratio, according to its first row, gives the coefficient of $\phi_r x$ the value of A_r as determined above. We observe that this ratio for R takes the indeterminate form $0/0$, when the a's approach a limiting fixed value a. In order to evaluate the limiting value of this ratio as the a's approach the limit a, we apply to the numerator and denominator the operator

$$\left(\frac{d}{da_2}\right)'_{a_2 = a} \ldots \left(\frac{d}{da_n}\right)^{n-1}_{a_n = a},$$

obtaining

$$\frac{\begin{vmatrix} fx & 1 & \phi_1 x & \ldots & \phi_{n-1} x \\ fa & 1 & \phi_1 a & \ldots & \phi_{n-1} a \\ f'a & 0 & \phi_1' a & \ldots & \phi'_{n-1} a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1} a & 0 & \phi_1^{n-1} a & \ldots & \phi^{n-1}{}_{n-1} a \end{vmatrix}}{\begin{vmatrix} \phi_1' a & \ldots & \phi'_{n-1} a \\ \cdot & \cdot & \cdot \\ \phi_1^{n-1} a & \ldots & \phi^{n-1}{}_{n-1} a \end{vmatrix}} = R. \quad (10)$$

It is to be distinctly observed that in this process we do not require the functions fx and $\sum\limits_0^{n-1} A_r\phi_r x$ to have a contact of the $(n-1)$th order at $x = a$ in order that we may equate their first $n - 1$ derivatives when $x = a$. What we require is merely that the functions fx and $\varphi_r x$ $(r = 1 \ldots n - 1)$ shall each have a determinate derivative at $x = a$, up to the $(n-1)$th operation. Of course, if fx and $\sum\limits_0^{n-1} A_r\phi_r x$ have an $(n-1)$th contact at $x = a$, then our value for R holds true as well; but it is not dependent on such a relation: it simply includes it.

If now the successive functions $\phi_r x$ $(r = 1 \ . \ . \ . \ n)$ may be formed in succession indefinitely according to a given law so that we may make r in $\phi_r x$ as great as we choose, then if it can be shown that R has for its limit zero, as r becomes infinite and at the same time the A's have limiting values such that $\sum\limits^{\infty} A_r\phi_r x$ is a converging series, then we may write

$$fx = A_0 + A_1\phi_1 x + A_2\phi_2 x + \ldots \text{ ad. inf.}$$

The value of R has been shown to be

$$\frac{\begin{vmatrix} 1, & \phi_1 x & \ldots & \phi_n x \\ 1, & \phi_1 a & \ldots & \phi_n a \\ 0, & \phi_1' a & \ldots & \phi_n' a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \phi_1^{n-1} a & \ldots & \phi_n^{n-1} a \end{vmatrix} \left(\dfrac{d}{dx}\right)^n_{x=u}}{\begin{vmatrix} \phi_1' a & \ldots & \phi_{n-1}' a \\ \cdot & \cdot & \cdot \\ \phi_1^{n-1} a & \ldots & \phi_{n-1}^{n-1} a \end{vmatrix} \left(\dfrac{d}{dx}\right)^n_{x=u}} \; \frac{\begin{vmatrix} fx, & 1, & \phi_1 x & \ldots & \phi_{n-1} x \\ fa, & 1, & \phi_1 a & \ldots & \phi_{n-1} a \\ f'a, & 0, & \phi_1' a & \ldots & \phi_{n-1}' a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}a, & 0, & \phi_1^{n-1} a & \ldots & \phi_{n-1}^{n-1} a \end{vmatrix}}{\begin{vmatrix} 1, & \phi_1 x & \ldots & \phi_n x \\ 1, & \phi_1 a & \ldots & \phi_n a \\ 0, & \phi_1' a & \ldots & \phi_n' a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \phi_1^{n-1} a & \ldots & \phi_n^{n-1} a \end{vmatrix}} \qquad (11)$$

in which u is some unknown value of x lying between x and a.

ON THE EARLY HISTORY OF THE NON-EUCLIDIAN GEOMETRY.

BY EMORY McCLINTOCK, LL.D.

It has until recently been supposed that the earliest work on non-euclidian geometry was Lobatschewsky's.* A much earlier production (1733) has been brought into notice by

* See BULLETIN of November, 1892, vol. II, No. 2, "On the Non-Euclidian Geometry."