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ON THE INJECTIVITY RADIUS GROWTH OF COMPLETE NONCOMPACT RIEMANNIAN MANIFOLDS*

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Abstract. In this paper we introduce a global geometric invariant $\alpha(M)$ related to injectivity radius to complete non-compact Riemannian manifolds and prove: If $\alpha(M^n) > 1$, then M^n is isometric to $\mathbb{R}^n$ when Ricci curvature is non-negative, and is diffeomorphic to $\mathbb{R}^n$ for $n \neq 4$ and homeomorphic to $\mathbb{R}^4$ for $n = 4$ if without any curved assumption. .

Key words. Injectivity radius, complete non-compact manifold.

AMS subject classifications. Primary 53C20; Secondary 53C35.

1. Introduction. The injectivity radius estimate plays an important role in the studying of global Riemannian geometry. For instance, see Klingenberg [8] and Cheeger [1]. But most work involves the injectivity radiuses of compact manifolds. Partial reason is that the injectivity radius of a compact manifold M

$$\mathrm{injrad}(M) = \min\{\mathrm{injrad}(p), p \in M\}$$

is always finite and positive. When the manifold is non-compact, we cannot say much about it.

In order to study complete non-compact Riemannian manifolds, we usually consider some objects involving infinity. Such as volume growth, Busemann function [9] (roughly speeking, a distance function from ∞) etc. In present paper we shall research the relationship between geometry and topology of complete non-compact Riemannian manifolds and the asymptotic properties of injectivity radiuses at infinity.

Let M be a complete Riemannian manifold. For a point $p \in M$, we denote the distance from p to x by $d(p, x)$. Recall that the injectivity radius of a point $p \in M$ is defined by

$$\mathrm{injrad}(p) := \sup\{r | \exp_p : B(0, r) \to B(p, r) \text{ is a diffeomorphism}\},$$

where $B(0, r)$ and $B(p, r)$ denote the open ball of radius r and center at $0 \in T_pM$ and $p \in M$.

We define the *injectivity radius growth* by

$$\alpha(M) = \varliminf_{r \to \infty} \frac{\mathrm{injrad}(p, r)}{r}, \tag{1}$$

where $\mathrm{injrad}(p, r) = \inf \{\mathrm{injrad}(x)|x \in M, d(p, x) = r\}$. We can show that $\alpha(M)$ is well-defined, i.e., it is independent on the choice of $p \in M$ (see proposition 2.1).

Our first theorem can be stated as follows.

THEOREM 1.1. *Let M^n be a complete non-compact Riemannian manifold with non-negative Ricci curvature. If $\alpha(M)$ defined by (1) satisfies $\alpha(M) > 1$, then M^n is isometric to $\mathbb{R}^n$.*

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Roughly speaking, theorem 1.1 says that if the injectivity radius at infinity is large enough, then a complete non-negative Ricci curved Riemannian manifold must be isometric to $\mathbb{R}^n$.

Without assumption of non-negative Ricci curvature in theorem 1.1, we have

THEOREM 1.2. *Let M^n be a complete non-compact Riemannian manifold. If $\alpha(M)$ defined by (1) satisfying $\alpha(M) > 1$, then M^n is diffeomorphic to $\mathbb{R}^n$ for $n \neq 4$ and homeomorphic to $\mathbb{R}^4$ for $n = 4$.*

The proof of theorem 1.2 lies on some deep topological results. In fact we prove that if $\alpha(M) > 1$, then the manifold must be contractible and simple connected at infinity. We don't know whether one has a purely geometric method. We also don't know whether M^n is diffeomorphic to $\mathbb{R}^4$ for $n = 4$.

REMARK 1.3. The $\alpha(M) > 1$ in theorem 1.1 and 1.2 is best possible. If $\alpha(M) \leq 1$, then we can construct counterexamples to theorem 1.1 and 1.2 (see section 5).

The rest of the paper is organized as follows: In section 2, we prove that $\alpha(M)$ is independent on the choice of point; We will give the proof of theorem 1.1 (resp. theorem 1.2) in section 3 (resp. section 4). The last section contains some examples and questions.

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2. On the injectivity radius growth. Let M be a complete non-compact Riemannian manifold. For a point $p \in M$, we write

$$\alpha(p) = \varliminf_{r\to\infty} \frac{\mathrm{injrad}(p,r)}{r},$$

here $\mathrm{injrad}(p,r) = \inf\{\mathrm{injrad}(x) | x \in M, d(p,x) = r\}$.

PROPOSITION 2.1. *The $\alpha(p)$ is independent on the choice of p. So we can write it as $\alpha(M)$.*

Proof. Let p, q be any two points of M. $d(p,q) = l$.

Case 1: $\alpha(p) = \infty$.

By the definition of $\alpha(p)$, for any $m > 0$, there exists $r_0 > 0$ such that for all $x \in M \setminus B(p, r_0)$, one has

$$\mathrm{injrad}(x) \geq m r_1,$$

here $r_1 = d(p,x)$. Then

$$\frac{\mathrm{injrad}(q,r_2)}{r_2} \geq \frac{m r_1}{r_2} \geq \frac{m(r_2 - l)}{r_2}$$

for all x such that $r_2 = d(q,x) \geq l + r_0$. Hence

$$\alpha(q) = \varliminf_{r\to\infty} \frac{\mathrm{injrad}(q,r)}{r} \geq \lim_{r\to\infty} \frac{m(r-l)}{r} = m.$$

Since m is any positive number, we must have $\alpha(q) = \infty$.

Case 2: $\alpha(p) < \infty$. From case 1 one must have $\alpha(q) < \infty$.

By the definition of $\alpha(q)$, for any $\epsilon > 0$, there exists $r_0 > 0$ such that for all $x \in M \setminus B(q, r_0)$, one has

$$\mathrm{injrad}(x) \geq \mathrm{injrad}(q, r_2) \geq (\alpha(q) - \epsilon) r_2,$$

where $r_2 = d(q, x)$. Hence for all x such that $d(p, x) = r_1 \geq l + r_0$, we have

$$\frac{\mathrm{injrad}(p, r_1)}{r_1} \geq \frac{(\alpha(q) - \epsilon) r_2}{r_1} \geq \frac{(\alpha(q) - \epsilon)(r_1 - l)}{r_1}$$

when $\alpha(q) - \epsilon \geq 0$ and

$$\frac{\mathrm{injrad}(p, r_1)}{r_1} \geq \frac{(\alpha(q) - \epsilon) r_2}{r_1} \geq \frac{(\alpha(q) - \epsilon)(r_1 + l)}{r_1}$$

when $\alpha(q) - \epsilon < 0$. Thus

$$\alpha(p) = \varliminf_{r \to \infty} \frac{\mathrm{injrad}(p, r)}{r} \geq \alpha(q) - \epsilon.$$

Since ϵ is any positive real number, we get

$$\alpha(p) \geq \alpha(q).$$

Similarly we can get

$$\alpha(q) \geq \alpha(p).$$

So we have

$$\alpha(p) = \alpha(q).$$

□

Note that even the manifold is non-compact, $\alpha(M)$ may be equal to zero. The cylinder $S^1 \times \mathbb{R}$ is a simple example. Obviously the $\alpha(M)$ of a Cartan-Hadamard manifold is ∞.

3. A proof of Theorem 1.1. In order to prove theorem 1.1, we need two lemmas.

LEMMA 3.1. *Let M^n be a complete non-compact Riemannian manifold with non-negative Ricci curvature. If $\alpha(M) > 1$, then M^n is isometric to $N \times \mathbb{R}^1$, where N is a complete non-negative Ricci curved manifold.*

Proof. Let p be a point of M. Let $\gamma_0(t)$ ($t \in [0, +\infty)$) be a ray starting at p. Let $\gamma(t)$ ($-\infty < t < +\infty$) be a geodesic through p such that $\gamma_0^{'} = \gamma^{'}$ at p. We claim that $\gamma(t)$ is a line.

Argue by contradiction. Assume that there exists $p_1,\ p_2 \in \gamma(t)$ such that p_2 is a cut point of p_1. Then there is another geodesic $\sigma(t)$ from p_1 to p_2.

Since $\alpha(M) > 1$, by the definition, for any $0 < \varepsilon < \frac{\alpha(M)-1}{2}$, there exists r_0 such that for all $r > r_0$, we have

$$\mathrm{injrad}(p, r) \geq (\alpha(M) - \varepsilon) r > (1 + \varepsilon) r.^1$$

[1]If $\alpha(M) = \infty$, we still have $\mathrm{injrad}(p, r) > (1 + \varepsilon) r$ when $r > r_0$.

Hence for $r > \max\{\frac{\max\{d(p_1,p),d(p_2,p)\}}{\varepsilon}, r_0\}$, we can choose $q \in \gamma_0(t)$ such that

$$\text{injrad}(q) \geq \text{injrad}(p, r) > (1 + \varepsilon)r, \tag{2}$$

and

$$d(p, q) = r = \text{ the length of } \gamma_0(t) \text{ from } p \text{ to } q.$$

So we have

$$d(p_i, q) \leq d(p_i, p) + d(p, q) < \varepsilon r + r = (1 + \varepsilon)r, i = 1, 2. \tag{3}$$

Therefore, we can conclude that from (2) and (3) that

$$p_1, p_2 \in B(q, (1 + \varepsilon)r) \subset B(q, \text{injrad}(q)). \tag{4}$$

Without losing generality, we assume that $p_1 = \gamma(t_1), p_2 = \gamma(t_2)$ and $t_1 < t_2$. Let $\gamma_1(t)$ be the curve from p_1 to q such that

$$\gamma_1(t)|_{[p_1,p_2]} = \sigma(t)$$

and

$$\gamma_1(t)|_{[p_2,q]} = \gamma(t)|_{[p_2,q]}.$$

Smoothing $\gamma_1(t)$ at p_2, we can obtain a smooth curve which the length is shorter than the length of $\gamma(t)|_{[p_1,q]}$. This is contradict to (4). Hence the claim is true.

Combining with the Cheeger-Gromoll splitting theorem [3], we complete the proof of the lemma. □

Lemma 3.2. *The N in lemma 3.1 is non-compact.*

Proof. If N is compact, then for any $q \in M = N \times \mathbb{R}^1$, one has $\text{injrad}(q) \leq \text{diam}(N)$. Hence $\alpha(M) = 0$. We get a contradiction. □

Proof of Theorem 1.1. Since N is non-compact, M must contain another ray starting at p which is contained in N. Repeating the procedure of lemma 3.1, 3.2 and using Cheeger-Gromoll splitting theorem again, we have that M^n is isometric to $N^{'} \times \mathbb{R}^2$, $N^{'}$ is non-compact. Step by step, we can conclude that M^n is isometric to $\mathbb{R}^n$.

4. A proof of Theorem 1.2.

Lemma 4.1. *Let M be a complete non-compact Riemannian manifold. If $\alpha(M) > 1$, then for every compact set C (not need connected), we can find $q \in M$ such that $C \subset B(q, injrad(q))$.*

Proof. Let p be a point of M. Let $\gamma(t)$ be a ray starting at p. Similar to the proof of lemma 3.1. For any $0 < \varepsilon < \frac{\alpha(M)-1}{2}$, there exists r_0 such that for all $r > r_0$, we have

$$\text{injrad}(p, r) \geq (\alpha(M) - \varepsilon)r > (1 + \varepsilon)r.$$

Let $s = \max\{d(p, x)|x \in C\}$. For $r > max\{\frac{s}{\varepsilon}, r_0\}$, we can choose $q \in \gamma(t)$ such that

$$\text{injrad}(q) \geq \text{injrad}(p, r) > (1 + \varepsilon)r,$$

and

$$d(p,q) = r = \text{ the length of } \gamma(t) \text{ from } p \text{ to } q.$$

So for any $x \in C$, one has

$$d(q,x) \leq d(x,p) + d(p,q) < \varepsilon r + r = (1+\varepsilon)r.$$

Thus $C \subset B(q, \mathrm{injrad}(q))$. □

COROLLARY 4.2. *Let M be a complete non-compact Riemannian manifold. If $\alpha(M) > 1$, then M is contractible.*

Proof. We only need to showed that the homotopy group $\pi_i(M)$ is trivial for $i \geq 0$. Let $f : (S^i, s_0) \to (M, p)$ be an element of $\pi_i(M, p)$. By lemma 4.1, we know that $f(S^i)$ is contained in some $B(q, \mathrm{injrad}(q))$. Hence it is contractible in M. □

A topological space T is said to be *1-connected at infinity* [10]: If for each compact set C of T, there is a compact set D of T with $C \subset D \subset T$, such that $T \setminus D$ is 1-connected.

COROLLARY 4.3. *Let $M^n (n \geq 3)$ be a complete non-compact Riemannian manifold. If $\alpha(M) > 1$, then M is 1-connected at infinity.*

Proof. Let C be any compact set of M. By lemma 4.1, we can choose a compact ball $B(q,r)$ such that $B(q,r)$ is diffeomorphic to Euclid unit ball and $C \subset B(q,r)$. Since M is 1-connected, we know that $M \setminus B(q,r)$ is 1-connected. Hence M is 1-connected at infinity. □

To prove theorem 1.2, we need the following deep theorem.

THEOREM 4.4. *Let $M^n (n \geq 3)$ be a contractible open smooth manifold and 1-connected at infinity. Then M^n is diffeomorphic to $\mathbb{R}^n$ for $n \neq 4$ and homeomorphic to $\mathbb{R}^4$ for $n = 4$.*

The case $n \geq 5$ is due to Stallings [10]. For $n = 3$, it is a consequence of Perelman's solution to Poincare conjecture and a theorem of Edwards [5] (see the theorem 1 and the third paragraph of [5]). The case $n = 4$ is due to Freedman [6] (see corollary 1.2 of [6]). Since Donaldson [4] found a smooth 4-manifold which is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to $\mathbb{R}^4$, we cannot get that M^n is diffeomorphic to $\mathbb{R}^4$ for $n = 4$.

Proof of Theorem 1.2. For the dimension ≥ 3, it is a consequence of corollary 4.2, corollary 4.3 and theorem 4.4. It follows from the Riemann mapping theorem as dimension $= 2$.

5. Examples and discussions. Now we give examples to show that the $\alpha(M)$ in theorem 1.1 and 1.2 is best possible.

EXAMPLE 5.1. *Let*

$$x^2 + y^2 - (z\tan(\theta))^2 = 0, z \geq 0$$

be the cone in $\mathbb{R}^3$. Smoothing the original point $p = (0,0,0)$, we get a complete noncompact surface with non-negative Gauss curvature. Clearly it is not isometric to $\mathbb{R}^2$.

It is straightforward to compute that

$$\alpha(M) = \begin{cases} sin(\pi sin\theta), & if\ 0 < \theta < \frac{\pi}{6}; \\ 1, & if\ \frac{\pi}{6} \le \theta < \frac{\pi}{2}. \end{cases}$$

Example 5.2. *We glue the following two surfaces*

$$x^2 + y^2 - (z\tan(\theta))^2 = 0, z \ge \epsilon > 0$$

and

$$x^2 + y^2 - (z\tan(\theta))^2 = 0, z \le -\epsilon < 0$$

along their edges. It is a non-simple connected surface.

One also can easy to check that

$$\alpha(M) = \begin{cases} sin(\pi sin\theta), & if\ 0 < \theta < \frac{\pi}{6}; \\ 1, & if\ \frac{\pi}{6} \le \theta < \frac{\pi}{2}. \end{cases}$$

Finally we propose two interesting questions.

Question 1. For a complete non-compact manifold, can we prove that every geodesic is a line as long as $\alpha(M) > 1$?

Question 2. Determining the minimal $\alpha_0 \in (0, 1]$ such that for any complete non-compact Riemannian manifold with non-negative Ricci curvature if $\alpha(M) > \alpha_0$, then M^n is diffeomorphic to $\mathbb{R}^n$.

Let us compare with the following two classical theorems:

1) Cheng's maximal diameter theorem [2]: A complete Riemannian manifold with $Ric_M \ge n-1$ and $diam = \pi$ must be isometric to $S^n(1)$.

2) Grove-Shiohama's generalized sphere theorem [7]: A closed Riemannian manifold with $sec_M \ge 1$ and $diam > \frac{\pi}{2}$ is homeomorphic to a sphere.

Roughly speaking, theorem 1.1 is a non-compact analogue of Cheng's theorem. Question 2 is to seek a non-compact analogue of Grove-Shiohama theorem.

We hope that the $\alpha(M)$ gives more contributions to the research of complete non-compact Riemannian manifolds.

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