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SCALING OF POISSON SPHERES AND COMPACT LIE GROUPS*

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Abstract. For $n \geq 2$, we show that on the standard Poisson homogeneous space $\mathbb{S}^{2n-1}$ (including $SU(2) \approx \mathbb{S}^3$), there exists a Poisson scaling ϕ_λ at any scale $\lambda > 0$ that is smooth on each symplectic leaf and continuous globally. A generalization to the case of the standard Bruhat-Poisson compact simple Lie groups endowed with a stronger topology is also valid.

Key words. Poisson Lie groups, Bruhat-Poisson structure, covariant Poisson structure, homogeneous Poisson structure, scaling, deformation quantization, compact simple Lie groups.

AMS subject classifications. Primary: 53D17; Secondary: 17B37, 53D55.

Introduction. In connection with the modular automorphism groups [W3], Weinstein showed [W2] that there is no nontrivial smooth scaling (ϕ, λ), called dilation, of the standard Bruhat-Poisson structure π on $SU(2)$ (or the reduced Poisson structure on its homogeneous space $\mathbb{S}^2$), i.e. a diffeomorphism ϕ of $SU(2)$ and a scalar $\lambda \neq 0$ such that $\phi^*\pi = \lambda^{-1}\pi$, other than $(\phi, \lambda) = (\iota, -1)$ where $\iota(u) := u^{-1}$ is the inverse map on $SU(2)$. This result is then generalized to all compact groups with Bruhat-Poisson structure by J.-H. Lu [W2].

However a very important geometric structure of a Poisson manifold is its decomposition into (maximal) symplectic leaves [W1] of various dimensions in general, which form some kind of "singular" foliation. Even though such a symplectic foliation has a nice local Poisson product structure [W1], it is not a standard regular foliation with a clean smooth structure everywhere. Instead, the closure of a symplectic leaf may meet many symplectic leaves of lower dimensions, and there further degeneracies of the Poisson structure occur, rendering a weaker sense of smoothness. From this viewpoint, the global smoothness of a scaling or dilation seems too strong a requirement in general. In this paper, we show that if a scaling $\phi_\lambda \equiv (\phi, \lambda)$ is required to be only continuous on the whole manifold but smooth on each symplectic leaf of a Poisson manifold, then it exists for all $\lambda > 0$ on the Poisson homogeneous space $\mathbb{S}^{2n-1}$ of the Bruhat-Poisson $SU(n)$. A Liouville vector field generating ϕ_λ is explicitly computed for $SU(2)$. Furthermore, if a standard Bruhat-Poisson compact simple Lie group K is endowed with some stronger topology that is still compatible with the original differential structure on each symplectic leaf of K, then a leafwise smooth and globally continuous scaling ϕ_λ exists on K for all $\lambda > 0$.

In [Sh1], it is shown that the standard Bruhat-Poisson $SU(2)$ can be quantized by Weyl calculus along all of its symplectic leaves to construct a C*-algebraic deformation quantization of the Poisson structure and yield the C*-algebra $C\left(SU(2)_q\right)$ of quantum $SU(2)$. The construction essentially composes a standard Weyl quantization of $\mathbb{C}$ with ϕ_λ^* for a family of continuous scalings ϕ_λ of the Poisson $SU(2)$. This method of quantization is intrinsically of "leaf-preserving" type as opposed to the "group-preserving" type, and there is a no-go theorem saying that these two types of quantization are disjoint [Sh2, Sh3]. For general Bruhat-Poisson compact simple Lie groups K, faithful leaf-preserving deformation quantizations have not been

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constructed, while group-preserving ones have been found [N] for $SU(n)$. Whether composing a standard Weyl quantization of $\mathbb{C}^n$ with ϕ_λ^* for a family of continuous scalings ϕ_λ of the Poisson $SU(n)$ results in a deformation quantization that produces the algebra of quantum $SU(n)$ remains to be studied.

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1. Scaling of Poisson $SU(2)$**.** In this section, we study how the standard Bruhat-Poisson structure on $SU(2)$ can be scaled smoothly leafwise and continuously globally.

We call a family of homeomorphisms $\phi_t : M \to M$, $t > 0$, on a Poisson manifold (M, π) a scaling of the Poisson structure of M if ϕ_t is a diffeomorphism from each symplectic leaf of M onto itself with

$$((\phi_t)_* \pi)(x) := (D\phi_t)_{\phi_{-t}(x)} (\pi(\phi_{-t}(x))) = t\pi(x)$$

and $\phi_1(x) = x$ for each $x \in M$. A basic example is the scaling of the standard symplectic structure $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{C}$ given by the family of diffeomorphisms μ_t of $\mathbb{C}$ defined by

$$\mu_t(w) := \sqrt{t}w$$

for $t > 0$ and $w \in \mathbb{C}$.

Recall that a smooth vector field X (assumed to be complete in this paper for simplicity) is called a Liouville vector field if $[X, \pi] = -\pi$, which if exists, generates a smooth scaling $\phi_t := \alpha_{-X}(\ln t)$ of π, where $\alpha_{-X}(s)$ with $s \in \mathbb{R}$ denotes the integral flow generated by the vector field $-X$. For example, the smooth vector field

$$X : w \mapsto \frac{-1}{2} w$$

whose opposite $-X$ generates $\phi_t \equiv \alpha_{-X}(\ln t) = \mu_t$ is a Liouville vector field on the standard symplectic manifold $\mathbb{C}$. Generalizing this notion to fit our consideration of non-smooth scalings, we call a continuous vector field X on the Poisson manifold (M, π) a Liouville vector field if $X|_L \in \Gamma(TL)$ is a smooth (tangential) vector field on L and $[X|_L, \pi|_L] = -\pi|_L$ is valid for each symplectic leaf L of M.

By embedding $SU(2)$ into $M_{2\times 2}(\mathbb{C}) \cong \mathbb{C}^4$ in the canonical way, we can concretely identify the tangent space $T_u SU(2)$ of $SU(2)$ at any $u \in SU(2)$ with the left (multiplicative) translation

$$L_u \mathfrak{su}(2) \equiv u\mathfrak{su}(2) \subset M_{2\times 2}(\mathbb{C})$$

of the Lie algebra $\mathfrak{su}(2) \equiv T_e SU(2) \subset M_{2\times 2}(\mathbb{C})$ by the matrix u, where $e \equiv I_2$ is the unit element of $SU(2)$.

Recall that the standard multiplicative Bruhat-Poisson structure on $SU(2)$ [D1, LW1, VSo1] is generated by

$$r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \wedge^2 T_e SU(2) \equiv \wedge^2 \mathfrak{su}(2).$$

More precisely, the Poisson 2-tensor π of the Bruhat-Poisson $SU(2)$ at

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2),$$

with $a,b,c,d \in \mathbb{C}$ such that $\bar{a}=d$, $b=-\bar{c}$, and $|a|^2+|c|^2=1$, is given by

$$\begin{aligned}\pi_u &= L_u(r) - R_u(r) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \wedge \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} \wedge \begin{pmatrix} ib & ia \\ id & ic \end{pmatrix} - \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} \wedge \begin{pmatrix} ic & id \\ ia & ib \end{pmatrix}.\end{aligned}$$

In the following, we denote by

$$L_0 = \left\{ \begin{pmatrix} a & -c \\ c & \bar{a} \end{pmatrix} \in SU(2) : \ c = \sqrt{1-|a|^2} \text{ and } |a| < 1 \right\}$$

the basic symplectic leaf that plays a crucially important role in the study of Bruhat-Poisson $SU(2)$ [VSo1]. We also use the notation

$$P = P_n : u \in SU(n) \mapsto u(e_1) \in \mathbb{S}^{2n-1}$$

for the fibration projection map, which is a diffeomorphism when $n=2$, where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{C}^n$.

Proposition. *There exists a (continuous and leafwise smooth) scaling ϕ_t, $t>0$, of the standard Bruhat-Poisson structure of $SU(2)$, which is generated leafwise as $\alpha_{-X}(\ln t)$ by the opposite $-X$ of the (continuous and leafwise smooth) Liouville vector field*

$$\begin{aligned} X : u &\equiv \begin{pmatrix} re^{i\theta} & -\sqrt{1-r^2}e^{-i\eta} \\ \sqrt{1-r^2}e^{i\eta} & re^{-i\theta} \end{pmatrix} \in SU(2) \\ &\mapsto \begin{pmatrix} \frac{(1-r^2)\ln(1-r^2)}{2r}e^{i\theta} & \frac{\sqrt{1-r^2}\ln(1-r^2)}{2}e^{-i\eta} \\ \frac{-\sqrt{1-r^2}\ln(1-r^2)}{2}e^{i\eta} & \frac{(1-r^2)\ln(1-r^2)}{2r}e^{-i\theta} \end{pmatrix} \in T_uSU(2) \subset M_{2\times 2}(\mathbb{C}).\end{aligned}$$

on $(SU(2),\pi)$.

Proof. For $u \in \overline{L_0} \subset SU(2)$ with $a = x+iy$ (and $c \geq 0$), we have

$$\begin{aligned} P_*(\pi_u) &= \begin{pmatrix} c \\ -\bar{a} \end{pmatrix} \wedge \begin{pmatrix} -ic \\ i\bar{a} \end{pmatrix} - \begin{pmatrix} c \\ -a \end{pmatrix} \wedge \begin{pmatrix} ic \\ ia \end{pmatrix} \\ &= -2\begin{pmatrix} c \\ -x \end{pmatrix} \wedge \begin{pmatrix} ic \\ -y \end{pmatrix} = -2\begin{pmatrix} c \\ -\operatorname{Re}(a) \end{pmatrix} \wedge \begin{pmatrix} ic \\ -\operatorname{Im}(a) \end{pmatrix}.\end{aligned}$$

It is easy to see that $P_*(\pi_u) = 0$ (and hence $\pi_u = 0$) at $u = \operatorname{diag}(e^{i\theta}, e^{-i\theta}) \in U(1) \subset SU(2)$, and hence the subgroup $U(1)$ consists of 0-dimensional symplectic leaves of $SU(2)$. So the canonical (left) $U(1)$-action on $SU(2)$ keeps the Poisson structure invariant [LW1, VSo1].

Under the diffeomorphism P, the element $\operatorname{diag}(e^{i\theta}, e^{-i\theta}) \in U(1)$ acts on $\mathbb{S}^3 \approx SU(2)$ as $(a,c) \mapsto (e^{i\theta}a, e^{-i\theta}c)$. So we get, for

$$u = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \in SU(2)$$

in general,

$$P_*(\pi_u) = -2 \begin{pmatrix} \bar{c} \\ -\operatorname{Re}\left(\frac{ac}{|c|}\right)\frac{c}{|c|} \end{pmatrix} \wedge \begin{pmatrix} i\bar{c} \\ -\operatorname{Im}\left(\frac{ac}{|c|}\right)\frac{c}{|c|} \end{pmatrix}$$

which vanishes at $u \in U(1)$ (where we formally take $c/|c| := 1$ if $c = 0$).

First we perform a change of variables by the diffeomorphism

$$\psi : u \in L_0 \mapsto z = \frac{a}{c} = \frac{a}{\sqrt{1-|a|^2}} \in \mathbb{C},$$

and get

$$\psi_*(\pi_u) = \frac{-2}{|c|^2}(1 \wedge i) = -2\left(1+|z|^2\right)(1 \wedge i) \in \wedge^2 T_z\mathbb{C}.$$

Under another change of variables by the diffeomorphism

$$\tau : z = re^{i\theta} \in \mathbb{C} \mapsto w = \sqrt{2^{-1}\ln(1+r^2)}e^{-i\theta} \in \mathbb{C},$$

we get the standard symplectic 2-tensor $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{C}$, i.e.

$$\begin{aligned} \tau_*(\psi_*(\pi_u)) &= \tau_*\left(-2\left(1+r^2\right)\left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right)_z\right) = -2\tau_*\left(\left(1+r^2\right)\left(\frac{1}{r}\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta}\right)_z\right) \\ &= -2\left(\frac{1+r^2}{r}\right)\left(\frac{1}{2}\frac{r}{(1+r^2)\sqrt{2^{-1}\ln(1+r^2)}}\frac{\partial}{\partial r} \wedge \left(-\frac{\partial}{\partial \theta}\right)\right)_w \\ &= \left(\frac{1}{\sqrt{2^{-1}\ln(1+r^2)}}\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta}\right)_w = \left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right)_w \in \wedge^2 T_w\mathbb{C}. \end{aligned}$$

Under the transformation τ, the canonical smooth scaling $\mu_t : w \mapsto \sqrt{t}w$, $t > 0$, of the standard symplectic structure $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{C}$ is equivalent to a scaling

$$z = re^{i\theta} \in \mathbb{C} \mapsto \sqrt{(1+r^2)^t - 1}e^{i\theta} \in \mathbb{C}$$

of the Poisson structure $\psi_*\pi$ on $\mathbb{C}$, which in turn, gives rise to a scaling of π on L_0 defined by

$$\phi_t : \begin{pmatrix} re^{i\theta} & -\sqrt{1-r^2} \\ \sqrt{1-r^2} & re^{-i\theta} \end{pmatrix} \in L_0 \mapsto \begin{pmatrix} \sqrt{1-(1-r^2)^t}e^{i\theta} & -\sqrt{(1-r^2)^t} \\ \sqrt{(1-r^2)^t} & \sqrt{1-(1-r^2)^t}e^{-i\theta} \end{pmatrix} \in L_0.$$

Note that this formula for ϕ_t can also be continuously applied to $u \in U(1) = \partial(L_0)$ with $r = 1$ and yield $\phi_t(u) = u$ for $u \in U(1)$. Also note that by our construction, the map

$$a = re^{i\theta} \in \overline{\mathbb{D}} \mapsto a_t = \sqrt{1-(1-r^2)^t}e^{i\theta} \in \overline{\mathbb{D}}$$

is smooth on the open unit disk $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

The smooth Liouville vector field $w \mapsto \frac{-1}{2}w$ on $\mathbb{C}$ pulls back via the diffeomorphism

$$\tau \circ \psi : \begin{pmatrix} re^{i\theta} & -\sqrt{1-r^2} \\ \sqrt{1-r^2} & re^{-i\theta} \end{pmatrix} \in L_0 \mapsto \sqrt{\frac{-1}{2}\ln(1-r^2)}e^{-i\theta} \in \mathbb{C},$$

to the smooth Liouville vector field

$$\begin{aligned} X_0 : u &\equiv \begin{pmatrix} re^{i\theta} & -\sqrt{1-r^2} \\ \sqrt{1-r^2} & re^{-i\theta} \end{pmatrix} \in L_0 \\ &\mapsto \begin{pmatrix} \frac{(1-r^2)\ln(1-r^2)}{2r}e^{i\theta} & \frac{\sqrt{1-r^2}\ln(1-r^2)}{2} \\ \frac{-\sqrt{1-r^2}\ln(1-r^2)}{2} & \frac{(1-r^2)\ln(1-r^2)}{2r}e^{-i\theta} \end{pmatrix} \in T_uSU(2) \end{aligned}$$

on L_0, by differentiating the curve

$$\begin{aligned} t &\mapsto (\tau \circ \psi)^{-1}\left(e^{\frac{-1}{2}t}\sqrt{\frac{-1}{2}\ln(1-r^2)}e^{-i\theta}\right) \\ &= (\tau \circ \psi)^{-1}\left(\sqrt{\frac{-1}{2}e^{-t}\ln(1-r^2)}e^{-i\theta}\right) \\ &= (\tau \circ \psi)^{-1}\left(\sqrt{\frac{-1}{2}\ln\left[(1-r^2)^{e^{-t}}\right]}e^{-i\theta}\right) \\ &= \begin{pmatrix} \sqrt{1-(1-r^2)^{e^{-t}}}e^{i\theta} & -\sqrt{(1-r^2)^{e^{-t}}} \\ \sqrt{(1-r^2)^{e^{-t}}} & \sqrt{1-(1-r^2)^{e^{-t}}}e^{-i\theta} \end{pmatrix} \end{aligned}$$

at $t = 0$.

Using the $U(1)$-action on L_0 that preserves the Poisson structure, we can get smooth scalings on other symplectic leaves of $SU(2)$. Actually since the above scaling ϕ_t on $u \in \overline{L_0}$ identified with $re^{i\theta} \in \overline{\mathbb{D}}$ is along the radial direction, the scalings obtained in this way on all symplectic leaves can be described by one formula

$$\begin{aligned} \phi_t : u &= \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} = \begin{pmatrix} re^{i\theta} & -se^{-i\eta} \\ se^{i\eta} & re^{-i\theta} \end{pmatrix} \in SU(2) \mapsto \phi_t(u) \\ &= \begin{pmatrix} f_t(r)e^{i\theta} & -g_t(s)e^{-i\eta} \\ g_t(s)e^{i\eta} & f_t(r)e^{-i\theta} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1-(1-r^2)^t}e^{i\theta} & -s^te^{-i\eta} \\ s^te^{i\eta} & \sqrt{1-(1-r^2)^t}e^{-i\theta} \end{pmatrix} \in SU(2) \end{aligned}$$

for $t > 0$ with $s = \sqrt{1-r^2}$, where

$$f_t(r) = \sqrt{1-(1-r^2)^t}$$

and

$$g_t(s) = s^t$$

are continuous functions of $r, s \in [0,1]$ that vanish at $r, s = 0$. By our construction, ϕ_t is smooth on each symplectic leaf of $SU(2)$, and it is clearly continuous globally on $SU(2)$. So we get a scaling ϕ_t, $t > 0$, of the standard Bruhat-Poisson structure of $SU(2)$.

Similarly, using the $U(1)$-action on L_0, we get the smooth Liouville vector field X_0 on L_0 extended to a (continuous) Liouville vector field

$$X : u = \begin{pmatrix} re^{i\theta} & -\sqrt{1-r^2}e^{-i\eta} \\ \sqrt{1-r^2}e^{i\eta} & re^{-i\theta} \end{pmatrix} \in SU(2)$$
$$\mapsto \begin{pmatrix} \frac{(1-r^2)\ln(1-r^2)}{2r}e^{i\theta} & \frac{\sqrt{1-r^2}\ln(1-r^2)}{2}e^{-i\eta} \\ \frac{-\sqrt{1-r^2}\ln(1-r^2)}{2}e^{i\eta} & \frac{(1-r^2)\ln(1-r^2)}{2r}e^{-i\theta} \end{pmatrix} \in T_u SU(2)$$

on $(SU(2), \pi)$, which is smooth on $SU(2) \backslash U(1)$ and vanishes on $U(1)$, where we adopt the convention

$$\left(1-r^2\right)^\beta \left(\ln\left(1-r^2\right)\right)\Big|_{r=1} := \lim_{r \to 1-} \left(1-r^2\right)^\beta \left(\ln\left(1-r^2\right)\right) = 0$$

for any $\beta > 0$. □

2. $SU(n)$**-homogeneous Poisson** $\mathbb{S}^{2n-1}$**.** In this section, we find explicitly a scaling of the $SU(n)$-covariant [LW2] or $SU(n)$-homogeneous [D2, VSo2] space $\mathbb{S}^{2n-1}$ for all $n \geq 2$. We use I_n to denote the $n \times n$ identity matrix.

By Soibelman's result [So], the symplectic leaves of the Bruhat-Poisson $SU(n)$ are exactly products of $t \in \mathbb{T}^{n-1} \subset SU(n)$ with the leaves L_0 in the $n-1$ canonically embedded basic $SU(2)$'s arranged in various orders. More precisely, let $\iota_k : SU(2) \to SU(n)$ be the canonical Poisson embedding defined by

$$\iota_k(u) := I_{k-1} \oplus u \oplus I_{n-k-1}$$

a block diagonal matrix for $u \in SU(2)$, and fix the reduced expression

$$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 ... \sigma_{n-1}\sigma_{n-2} ... \sigma_2\sigma_1$$

of the maximal element in the Weyl group $\mathcal{W}_n$ of $SU(n)$ with respect to the Bruhat ordering [H, Sh4], where $\{\sigma_k\}_{k=1}^n$ are the reflections associated with the fundamental roots determined by the embeddings $\{\iota_k\}_{k=1}^n$.

Soibelman's classification of symplectic leaves of Bruhat-Poisson compact simple Lie groups [So] (cf. the next section for more details) implies that there is a one-to-one correspondence

$$(\delta, K) \leftrightarrow \delta L_K = \delta\iota_K\left((L_0)^m\right)$$

between symplectic leaves δL_K of the Bruhat-Poisson $SU(n)$ and pairs (δ, K) of a point $\delta \in \mathbb{T}^{n-1} \subset SU(n)$ and an "admissible" sequence $K = (k_1, k_2, ..., k_m)$, i.e. a concatenation $J_1J_2...J_n$ of sequences J_i which are either empty or equal to $(i, i-1, ..., i-k_i)$ for some $0 \leq k_i < i$, where

$$\iota_K : (u_1, ..., u_m) \in (L_0)^m \mapsto \iota_{k_1}(u_1) ... \iota_{k_m}(u_m) \in SU(n)$$

and $L_K := \iota_K\left((L_0)^m\right)$ which is set to be $\{I_n\}$ if K is an empty sequence.

Note that the multiplication map

$$\iota = \iota_{(n-1,\dots,1)} : (u_1, u_2, .., u_{n-1}) \in SU(2)^{n-1} \\ \mapsto \iota_{n-1}(u_1)\,\iota_{n-2}(u_2)\cdots\iota_1(u_{n-1}) \in SU(n),$$

is Poisson on each product of symplectic leaves, and the map

$$P_n : u \in SU(n) \to u(e_1) \in \mathbb{S}^{2n-1}$$

viewed as the $SU(n)$-action on $\mathbb{S}^{2n-1}$ restricted to $SU(n) \times \{e_1\}$ is Poisson since $\{e_1\}$ is a 0-dimensional leaf of $\mathbb{S}^{2n-1}$.

By induction, it is easy to verify that

$$(P_n \circ \iota)(u_1, u_2, .., u_{n-1}) = P_n(\iota_{n-1}(u_1)\,\iota_{n-2}(u_2)\cdots\iota_1(u_{n-1})) \\ = a_{n-1}e_1 + \sum_{k=2}^{n-1}(a_{n-k}c_{n-k+1}\cdots c_{n-1})\,e_k + (c_1c_2\cdots c_{n-1})\,e_n \in \mathbb{S}^{2n-1}$$

for

$$u_k = \begin{pmatrix} a_k & -c_k \\ c_k & \overline{a_k} \end{pmatrix} = \begin{pmatrix} r_k e^{i\theta_k} & -s_k e^{-i\eta_k} \\ s_k e^{i\eta_k} & r_k e^{-i\theta_k} \end{pmatrix} \in \delta_{\eta_k} L_0 \subset SU(2)$$

where

$$\delta_\eta := \mathrm{diag}\left(e^{-i\eta}, e^{i\eta}\right) \in U(1) = \mathbb{T}.$$

By taking $a_0 := 1$ and $a_{n-k}c_{n-k+1}\cdots c_{n-1} := a_{n-1}$ when $k = 1$, we can write more compactly

$$P_n \circ \iota : (u_1, u_2, .., u_{n-1}) \in SU(2)^{n-1} \mapsto z = \sum_{k=1}^{n}(a_{n-k}c_{n-k+1}\cdots c_{n-1})\,e_k \in \mathbb{S}^{2n-1}$$

from which it is not hard to check that $P_n \circ \iota$ is surjective. Actually the following lemma provides some more specific details.

LEMMA. The function $P_n \circ \iota$ restricted to

$$L_{m,\eta_1,\dots,\eta_{n-1}} := \left\{\left(\delta_{\eta_1}, ..., \delta_{\eta_{n-1-m}}\right)\right\} \times \left(\delta_{\eta_{n-m}} L_0\right) \times ... \times \left(\delta_{\eta_{n-1}} L_0\right),$$

with $\left(\delta_{\eta_1}, ..., \delta_{\eta_{n-1}}\right) \in \mathbb{T}^{n-1}$ and $1 \le m \le n-1$, is a (Poisson) diffeomorphism onto $\mathbb{S}^{2m}_\eta$, where

$$\eta := -\eta_{n-1-m} + \eta_{n-m} + \cdots + \eta_{n-1}$$

and

$$\mathbb{S}^{2m}_\eta := \mathbb{S}^{2n-1} \cap \left(\mathbb{C}^m \times e^{i\eta}\mathbb{R}_> \times \{0\}^{n-1-m}\right).$$

In particular, $P_n \circ \iota$ maps $SU(2)^{n-1}$ onto $\mathbb{S}^{2n-1}$ which is the disjoint union of $\mathbb{S}^{2m}_\eta$ with $0 \le m < n$ and $\eta \in [0, 2\pi)$.

Proof. Since $\iota\left(L_{m,\eta_1,\dots,\eta_{n-1}}\right)$ is a symplectic leaf of $SU(n)$ and P_n is a Poisson map, it suffices to show that $P_n \circ \iota$ restricted to $L_{m,\eta_1,\dots,\eta_{n-1}}$ is a diffeomorphism onto $\mathbb{S}^{2m}_\eta$ for $\eta := -\eta_{n-1-m} + \eta_{n-m} + \cdots + \eta_{n-1}$.

Note that the general condition $|a_j|^2 + |c_j|^2 = 1$ for all j implies for any $M < n$,

$$\begin{aligned}
&\sum_{k=1}^{M} |a_{n-k} (c_{n-k+1} \cdots c_{n-1})|^2 \\
&= 1 - |c_{n-1}|^2 + \sum_{k=2}^{M} \left(1 - |c_{n-k}|^2\right) |c_{n-k+1}|^2 \cdots |c_{n-1}|^2 \\
&= 1 - |c_{n-1}|^2 + \sum_{k=2}^{M} \left[|c_{n-k+1}|^2 \cdots |c_{n-1}|^2 - |c_{n-k}|^2 \cdots |c_{n-1}|^2\right] \\
&= 1 - |c_{n-M}|^2 \cdots |c_{n-1}|^2 .
\end{aligned}$$

For $z \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ and $(u_1, u_2, .., u_{n-1}) \in SU(2)^{n-1}$,

$$(u_1, u_2, .., u_{n-1}) \in L_{m,\eta_1,\ldots,\eta_{n-1}} \cap (P_n \circ \iota)^{-1} (z)$$

if and only if the conditions (i)

$$c_j = e^{i\eta_j} \sqrt{1 - |a_j|^2} \neq 0$$

for all $j \geq n - m$, (ii) $a_j = e^{-i\eta_j}$ for all $j < n - m$, and (iii) for all $1 \leq k \leq n - 1$,

$$a_{n-k} (c_{n-k+1} \cdots c_{n-1}) = z_k$$

(including $a_{n-1} = z_1$ when $k = 1$) are satisfied.

Note that conditions (i)-(iii) imply that

$$z_{m+1} = a_{n-m-1} (c_{n-m} \cdots c_{n-1}) \in e^{-\eta_{n-m-1}+\eta_{n-m}+\cdots+\eta_{n-1}} \mathbb{R}_>$$

where $\mathbb{R}_> := \{x \in \mathbb{R} : x > 0\}$, and $z_k = 0$ for all $k > m + 1$ since $c_{n-m-1} = 0$, or equivalently, $z \in \mathbb{S}^{2m}_\eta$. So $P_n \circ \iota$ maps $L_{m,\eta_1,\ldots,\eta_{n-1}}$ into $\mathbb{S}^{2m}_\eta$.

Also note that the condition (iii) implies

$$1 - \sum_{k=1}^{M} |z_k|^2 = 1 - \sum_{k=1}^{M} |a_{n-k} (c_{n-k+1} \cdots c_{n-1})|^2 = |c_{n-M}|^2 \cdots |c_{n-1}|^2 .$$

For $z \in \mathbb{S}^{2m}_\eta$, we have $|z_k| < 1$ for all $k \leq m$ and $z_k = 0$ for all $k > m + 1$, and $1 - \sum_{k=1}^{M} |z_k|^2 = 0$ if and only if $M \geq m + 1$. So under the condition (iii), we get $c_{n-k} \neq 0$ (and hence $u_{n-k} \in \mathbb{T}L_0$) for all $k < m + 1$ and $c_{n-m-1} = 0$ (and hence $|a_{n-m-1}| = 1$), which then imply that for all $k \leq m + 1$,

$$a_{n-k} = z_k (c_{n-k+1} \cdots c_{n-1})^{-1}$$

is uniquely well-defined, and furthermore

$$|a_{n-m-1}| = \left|z_{m+1} (c_{n-m} \cdots c_{n-1})^{-1}\right| = \sqrt{\frac{|z_{m+1}|^2}{|c_{n-m}|^2 \cdots |c_{n-1}|^2}} = \sqrt{\frac{|z_{m+1}|^2}{1 - \sum_{k=1}^{m} |z_k|^2}} = 1$$

which combined with condition (ii) gives $a_{n-m-1} := e^{-i\eta_{n-m-1}}$. For all $k > m + 1$, we see that with $z_k = 0$ and $c_{n-k+1} \cdots c_{n-1} = 0$, $a_{n-k} = e^{-i\eta_{n-k}}$ is the unique solution for both conditions (ii) and (iii). Thus for any $z \in \mathbb{S}^{2m}_{\eta}$, there is a unique $(u_1, u_2, .., u_{n-1}) \in L_{m,\eta_1,...,\eta_{n-1}}$ such that $(P_n \circ \iota)(u_1, u_2, .., u_{n-1}) = z$.

So $P_n \circ \iota$ is a bijective smooth map from $L_{m,\eta_1,...,\eta_{n-1}}$ to $\mathbb{S}^{2m}_{\eta}$, whose inverse is also a smooth map $z \mapsto (u_1, u_2, .., u_{n-1})$ given by the formulas

$$a_{n-k} := \begin{cases} z_k \left(c_{n-k+1} \cdots c_{n-1}\right)^{-1}, & \text{if} \quad k \le m+1 \\ e^{-i\eta_{n-k}}, & \text{if} \quad k > m+1 \end{cases}$$

and

$$c_j = e^{i\eta_j} \sqrt{1 - |a_j|^2}$$

for all j. □

Theorem. *There exists a (continuous and leafwise smooth) scaling ψ_t, $t > 0$, of the standard Bruhat-Poisson structure of $\mathbb{S}^{2n-1}$ for all $n \ge 2$.*

Proof. Let ϕ_t be the scaling of $SU(2)$ obtained in the previous proposition. The scaling $(\phi_t)^{n-1}$ of the product Poisson manifold $SU(2)^{n-1}$ restricted to the symplectic leaf $L_{m,\eta_1,...,\eta_{n-1}}$ induces, under the diffeomorphism $P_n \circ \iota$, a corresponding smooth scaling of $\mathbb{S}^{2m}_{\eta}$ with $\eta := -\eta_{n-1-m} + \eta_{n-m} + \cdots + \eta_{n-1}$, given by

$$\left(\psi_{m,\eta_1,...,\eta_{n-1}}\right)_t : \left(r_{n-1} e^{i\theta_{n-1}}\right) e_1 + \sum_{k=2}^{m+1} \left(r_{n-k} s_{n-k+1} \cdots s_{n-1} e^{i\left(\theta_{n-k} + \eta_{n-k+1} + \cdots + \eta_{n-1}\right)}\right) e_k$$

$$\mapsto \left[f_t(r_{n-1}) e^{i\theta_{n-1}}\right] e_1 + \sum_{k=2}^{m+1} \left[f_t(r_{n-k}) g_t(s_{n-k+1}) \cdots g_t(s_{n-1}) e^{i\left(\theta_{n-k} + \eta_{n-k+1} + \cdots + \eta_{n-1}\right)}\right] e_k$$

for $m \le n-1$ with $r_{n-1-m} = 1$ and $\theta_{n-1-m} = -\eta_{n-1-m}$, where $s_j = \sqrt{1 - r_j^2}$ for all j, and as before, we take $r_0 = a_0 := 1$ when $m = n-1$.

Since the scaling $\left(\psi_{m,\eta_1,...,\eta_{n-1}}\right)_t$ of $\mathbb{S}^{2m}_{\eta}$ keeps the angle (argument) of each complex coefficient invariant, it depends only on η and can be written as

$$\begin{aligned} (\psi_\eta)_t : z = \left(r_{n-1} e^{i\beta_1}\right) e_1 + \sum_{k=2}^{m+1} \left(r_{n-k} s_{n-k+1} \cdots s_{n-1} e^{i\beta_k}\right) e_k \in \mathbb{S}^{2m}_{\eta} \mapsto \\ \left(f_t(r_{n-1}) e^{i\beta_1}\right) e_1 + \sum_{k=2}^{m+1} \left(f_t(r_{n-k}) g_t(s_{n-k+1}) ... g_t(s_{n-1}) e^{i\beta_k}\right) e_k \in \mathbb{S}^{2m}_{\eta} \end{aligned}$$

with $r_{n-1-m} = 1$ and $\theta_{n-1-m} = -\eta_{n-1-m}$.

Since $\mathbb{S}^{2n-1}$ is a disjoint union of $\mathbb{S}^{2m}_{\eta}$ with $0 \le m < n$ and $\eta \in [0, 2\pi)$, we get a well-defined function $\psi_t : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ whose restriction to each symplectic leaf $\mathbb{S}^{2m}_{\eta}$ is the smooth scaling $(\psi_\eta)_t$. Now it remains to show that ψ_t is a homeomorphism of $\mathbb{S}^{2n-1}$.

Note that if $|a_{n-j}| = 1$ for some j, i.e. $u_{n-j} \in U(1) = \mathbb{T}$ is a diagonal 2×2 matrix, then

$$\begin{aligned} (P_n \circ \iota)(u_1, .., u_{n-1}) &= \iota_{n-1}(u_1) \cdots \iota_1(u_{n-1})(e_1) \\ &= \iota_j(u_{n-j}) \cdots \iota_1(u_{n-1})(e_1) \in \mathbb{C}^j \times \{0\} \subset \mathbb{C}^n. \end{aligned}$$

So $(P_n \circ \iota)(u_1, ..., u_{n-1}) \in \mathbb{S}^{2m}_{\eta}$ if and only if $|a_{n-m-1}| = 1 > |a_k|$ for all $k \ge n-m$, which implies that

$$(*) \qquad (P_n \circ \iota)(u_1, ..., u_{n-1}) = (P_n \circ \iota)(I_2, ..., I_2, u_{n-m-1}, ..., u_{n-1})$$

with $u_k \in \mathbb{T}L_0$ for all $k \geq n-m$.

Now $SU(2)^{n-1}$ is a disjoint union of symplectic leaves $F = \prod_{k=1}^{n-1} A_k$ with A_k either a singleton in $\mathbb{T} \subset SU(2)$ or a disk $e^{i\eta_k} L_0 \subset SU(2)$ for some η_k. If m is the largest index with A_{n-m-1} a singleton, then $\prod_{k=1}^{n-1} \{I_2\} \times \prod_{k=n-m-1}^{n-1} A_k$ equals some $L_{m,\eta_1,\ldots,\eta_{n-1}}$ and by (*),

$$(P_n \circ \iota)(F) = (P_n \circ \iota)\left(L_{m,\eta_1,\ldots,\eta_{n-1}}\right) = \mathbb{S}^{2m}_{\eta}.$$

It is not hard to see that the equality

$$\psi_t \circ (P_n \circ \iota) = (P_n \circ \iota) \circ (\phi_t)^{n-1},$$

clearly valid on $L_{m,\eta_1,\ldots,\eta_{n-1}}$, is also valid on $(P_n \circ \iota)(F)$ because of (*), for each symplectic leaf F of $SU(2)^{n-1}$.

Thus we have the commuting diagram

$$\begin{array}{ccc} SU(2)^{n-1} & \overset{(\phi_t)^{n-1}}{\rightarrow} & SU(2)^{n-1} \\ \downarrow_{P_n \circ \iota} & \circlearrowleft & \downarrow_{P_n \circ \iota} \\ \mathbb{S}^{2n-1} & \overset{\psi_t}{\rightarrow} & \mathbb{S}^{2n-1} \end{array}$$

where $\mathbb{S}^{2n-1}$ with its standard topology can be viewed as a quotient topological space of the compact Hausdorff space $SU(2)^{n-1}$ with $P_n \circ \iota$ as the quotient map, and ψ_t can be viewed as a well-defined map on $\mathbb{S}^{2n-1}$ induced by the continuous map $(\phi_t)^{n-1}$ on $SU(2)^{n-1}$. It is easy to see that the map ψ_t on $\mathbb{S}^{2n-1}$ is continuous.

So we have a well-defined continuous and leafwise smooth scaling ψ_t, $t > 0$, on $\mathbb{S}^{2n-1}$. □

We remark that ψ_t can be described by the formula

$$\psi_t : z = \left[r_{n-1} e^{i\beta_1}\right] e_1 + \sum_{k=2}^{n-1} \left[r_{n-k} s_{n-k+1} \ldots s_{n-1} e^{i\beta_k}\right] e_k + \left[s_1 s_2 \ldots s_{n-1} e^{i\beta_n}\right] e_n \in \mathbb{S}^{2n-1} \mapsto$$

$$\psi_t(z) = \left[f_t(r_{n-1}) e^{i\beta_1}\right] e_1 + \sum_{k=2}^{n-1} \left[f_t(r_{n-k}) g_t(s_{n-k+1}) \ldots g_t(s_{n-1}) e^{i\beta_k}\right] e_k$$
$$+ \left[g_t(s_1) g_t(s_2) \ldots g_t(s_{n-1}) e^{i\beta_n}\right] e_n \in \mathbb{S}^{2n-1}$$

with $s_j = \sqrt{1 - r_j^2}$ for all j.

3. Bruhat-Poisson compact simple Lie groups. In this section, we use the scaling ϕ_t of Bruhat-Poisson $SU(2)$ to construct a scaling for the standard Bruhat-Poisson compact simple Lie groups K with a topology stronger than the standard one but still compatible with the original differential structure on each symplectic leaf.

For a simple complex Lie group G, we fix a root system Λ with (positive) simple roots $\{\alpha_i\}_{i=1}^r$ for its Lie algebra $\mathfrak{g}$ and a corresponding Cartan-Weyl basis $\{X_\alpha\}_{\alpha \in \Lambda} \cup \{H_i\}_{i=1}^r$ with $H_i = [X_{\alpha_i}, X_{-\alpha_i}]$ for each i. The real form (i.e. the $+1$-eigenspace) for the antilinear involution $\omega : \mathfrak{g} \to \mathfrak{g}$ defined by $\omega(X_\alpha) = -X_{-\alpha}$ and $\omega(H_i) = -H_i$ for all $\alpha \in \Lambda$ and $1 \leq i \leq r$ is the Lie algebra $\mathfrak{g}$ of a maximal compact subgroup K of G. We consider only K endowed with the standard Bruhat-Poisson structure generated by the tensor

$$\mathbf{r} = \frac{i}{2} \sum_{\alpha \in \Lambda_+} (X_{-\alpha} \otimes X_\alpha - X_\alpha \otimes X_{-\alpha}) \in \mathfrak{k} \wedge \mathfrak{k}.$$

There is a well-known canonical Poisson embedding $\iota_{i_*} : SU(2) \to K$ for each basic triple $\{X_{\alpha_i}, X_{-\alpha_i}, H_i\}$, $1 \le i \le r$.

Recall that the Weyl group W of K is a Coxeter group [H] generated by $\{\sigma_i\}_{i=1}^r$ with $(\sigma_i\sigma_j)^{m_{ij}} = 1$ for $m_{ii} = 1$ and some $m_{ij} \in \{2,3,4,6\}$ if $i \ne j$, where $\sigma_i = \sigma_{\alpha_i}$ is the reflection on the dual $\mathfrak{h}^*$ of the Lie subalgebra $\mathfrak{h} := \mathrm{Span}\{H_i\}_{i=1}^r$ determined by the root α_i. If $w = \sigma_{i_1}\sigma_{i_2}...\sigma_{i_m}$ is the shortest expansion of w in σ_i's, then $\sigma_{i_1}\sigma_{i_2}...\sigma_{i_m}$ is called a reduced expression for w and $\ell(w) := m$ is the length of w. The Bruhat ordering on W is the partial ordering generated by the relations $w_1 < w_2$ satisfying $\sigma_\alpha w_1 = w_2$ and $\ell(w_1) + 1 = \ell(w_2)$ for some simple root α. It is known that there is a unique maximal element $\tilde{w} = \sigma_{l_1}\sigma_{l_2}...\sigma_{l_M}$ in W with respect to the Bruhat ordering [H] and every element of W has a reduced expression embedded in the expression $\sigma_{l_1}\sigma_{l_2}...\sigma_{l_M}$ [BB] (i.e. obtainable by removing some σ_{l_j}'s from $\tilde{w}$).

It is an interesting discovery [So] that the symplectic leaves L of K are in one-to-one correspondence with elements (δ, w) of $\mathbb{T}^r \times W$ and hence with the irreducible *-representations π_L of the algebra $C(K_q)^\infty$ of regular functions of a quantum group K_q. More explicitly, for each $(\delta, w) \in \mathbb{T}^r \times W$, we fix a reduced expression $\sigma_{i_1}\sigma_{i_2}...\sigma_{i_m}$ for $w \in W$ such that $\sigma_{i_1}\sigma_{i_2}...\sigma_{i_m}$ is embedded in $\sigma_{l_1}\sigma_{l_2}...\sigma_{l_M}$, and then the set $\delta L_w \subset K$ is the corresponding symplectic leaf, where $L_w := \iota_w\left((L_0)^m\right)$ and

$$\iota_w : (u_1, ..., u_m) \in \left(\overline{L_0}\right)^m \mapsto \iota_{i_1}(u_1)...\iota_{i_m}(u_m) \in K.$$

With $w = \sigma_{i_1}\sigma_{i_2}...\sigma_{i_m}$ embedded in $\sigma_{l_1}\sigma_{l_2}...\sigma_{l_M}$, we have $i_k = l_{j_k}$ for $k \le m$ where

$$1 \le j_1 < j_2 < ... < j_m \le M.$$

We define

$$\mathcal{L}_w := \mathbb{T}^r \times \left\{u \in \left(\overline{L_0}\right)^M \mid u_{j_k} \in L_0 \text{ for } k \le \ell(w) \text{ and } u_j = I_2 \text{ for other } j\text{'s}\right\}.$$

Let $\mathcal{L} \subset \mathbb{T}^r \times \left(\overline{L_0}\right)^M$ be the union of these disjoint subsets $\mathcal{L}_w$ with $w \in W$. Then the continuous map

$$\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}} : (\delta, u_1, ..., u_M) \in \mathbb{T}^r \times \left(\overline{L_0}\right)^M \mapsto \delta\iota_{l_1}(u_1)...\iota_{l_M}(u_M) \in K$$

sends $\mathcal{L}$ onto K. By viewing K as a quotient space of $\mathcal{L}$, we get a quotient topology $\mathcal{T}$ on K from $\mathcal{L}$ via the map $(\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}})|_{\mathcal{L}}$. By definition, $\mathcal{T}$ consists of sets $A \subset K$ with $(\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}})^{-1}(A)$ open in $\mathcal{L}$, and is stronger than (i.e. contains) the original topology on K and hence still Hausdorff. Furthermore, $\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}}$ is homeomorphic on each $\mathcal{L}_w$ and hence the topology $\mathcal{T}$ is compatible with the original differential structure on each symplectic leaf of K.

THEOREM. *There exists a (continuous and leafwise smooth) scaling Φ_t, $t > 0$, of the standard Bruhat-Poisson structure of a compact simple Lie group K when K is endowed with the topology $\mathcal{T}$.*

Proof. Since the Bruhat-Poisson structure on K is multiplicative and $\mathbb{T}^r$ consists of 0-dimensional leaves $\{\delta\}$ whose action by multiplication preserves the Bruhat-Poisson structure, the diffeomorphic map

$$\iota_{\delta,w} : (u_1, u_2, .., u_m) \in (L_0)^m \mapsto \delta\iota_w(u_1, u_2, .., u_m) \in \delta L_w$$

for $\delta \in \mathbb{T}^r$ and $w = \sigma_{i_1}\sigma_{i_2}...\sigma_{i_m}$ is symplectic. So the smooth scaling

$$(\phi_t)^m(u_1, u_2, .., u_m) = (\phi_t(u_1), \phi_t(u_2), .., \phi_t(u_m))$$

of the symplectic product space $(L_0)^m$ induces, via $\iota_{\delta,w}$, a smooth scaling $(\Phi_{\delta,w})_t$ of the symplectic leaf δL_w.

Since K is the disjoint union of its symplectic leaves δL_w [W1], we get a family of well-defined functions $\Phi_t : K \to K$ which is the diffeomorphism $(\Phi_{\delta,w})_t$ on each symplectic leaf δL_w. It remains to show that Φ_t is continuous on K with respect to the topology $\mathcal{T}$.

By restricting to $\mathcal{L}$, we get a continuous map

$$\left(\mathrm{id}_{\mathbb{T}^{n-1}} \times (\phi_t)^M\right)\Big|_{\mathcal{L}} : \mathcal{L} \subset \mathbb{T}^r \times \left(\overline{L_0}\right)^M \to \mathcal{L} \subset \mathbb{T}^r \times \left(\overline{L_0}\right)^M$$

on $\mathcal{L}$. It is easy to see that $(\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}})(\mathcal{L}_w) = \mathbb{T}^r L_w$ for all $w \in W$, and

$$\Phi_t \circ (\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}})|_{\mathcal{L}} = (\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}})|_{\mathcal{L}} \circ \left(\mathrm{id}_{\mathbb{T}^{n-1}} \times (\phi_t)^M\Big|_{\mathcal{L}}\right).$$

So when K is endowed with the quotient topology $\mathcal{T}$ via $(\mathrm{id}_{\mathbb{T}^r} \times \iota_{\tilde{w}})|_{\mathcal{L}}$, the continuity of $\left(\mathrm{id}_{\mathbb{T}^{n-1}} \times (\phi_t)^M\right)\Big|_{\mathcal{L}}$ on $\mathcal{L}$ implies the continuity of Φ_t on K. ∎

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