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On the Diophantine Equation $x^2 - kxy + ky^2 + ly = 0$, $l \in \{1,2,4,8\}$

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Abstract

In this study, we consider the Diophantine equation $x^2 - kxy + ky^2 + ly = 0$, $l \in \{1,2,4,8\}$ and determine the values of k when the equation has infinitely many positive integer solutions x and y.

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1 Introduction

A Diophantine equation is an indeterminate polynomial equation in which only integer solutions are allowed. Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. Diophantine equations get their name from Hellenistic mathematician Diophantus of Alexandria who is best known for his Arithmetica, a work on the solution of algebraic equations and on the theory of numbers. In general, the Diophantine equation of degree 2 is an equation given by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{1.1}$$

Any Diophantine equation of the form $x^2 - dy^2 = N$ is known as Pell equation, where d is not a perfect square, x and y are integers, and N is any nonzero fixed integer. Pell equation is a special case of (1.1). Also for $N = \pm 1$, the equation $x^2 - dy^2 = \pm 1$ is known as classical Pell equation. The Pell equation is perhaps the oldest Diophantine equation that has interested mathematicians all over the world for probably more than a 1000 years now. The name of this equation arose from Leonhard Euler's mistakenly attributing its study to John Pell, who searched for integer solutions to equations of this type in 17-th century. The notations

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(x,y) and $x + y\sqrt{d}$ are used interchangeably to denote solutions to the equation $x^2 - dy^2 = N$. If x and y are both positive, then we say that $x + y\sqrt{d}$ is positive solution to the equation $x^2 - dy^2 = N$. The least positive integer solution $x_1 + y_1\sqrt{d}$ to this equation is called the fundamental solution. If $x_1 + y_1\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = -1$, then it is well known that $(x_1 + y_1\sqrt{d})^2$ is the fundamental solution to the equation $x^2 - dy^2 = 1$. Moreover, if $x_1 + y_1\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = 1$, then all other positive integer solutions to this equation are given by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

with $n \geq 1$. If $x + y\sqrt{d}$ is a solution of the equation $x^2 - dy^2 = N$ and $u + v\sqrt{d}$ is a solution of the equation $x^2 - dy^2 = 1$, then $(u + v\sqrt{d})(x + y\sqrt{d}) = (ux + dvy) + (uy + vx)\sqrt{d}$ is also a solution of the equation $x^2 - dy^2 = N$. This means that if the equation $x^2 - dy^2 = N$ has a solution, then it has infinitely many solutions.

For further details on Diophantine and Pell equations we refer the reader to [1, 2, 3, 4, 5, 6, 7].

In [12], Marlewski and Zarzycki considered the Diophantine equation

$$x^2 - kxy + y^2 + x = 0 \tag{1.2}$$

and they proved that Eq.(1.2) has infinitely many positive integer solutions x and y if and only if $k = 3$ but the same equation has no positive integer solutions x and y when $k > 3$.

In [13], Yuan and Hu considered the Diophantine equations

$$x^2 - kxy + y^2 + 2x = 0 \tag{1.3}$$

and

$$x^2 - kxy + y^2 + 4x = 0. \tag{1.4}$$

They showed that Eq.(1.3) has infinitely many positive integer solutions x and y if and only if $k = 3,4$ and Eq.(1.4) has infinitely many positive integer solutions x and y if and only if $k = 3,4,6$.

The main purpose of the present paper is to determine when the following four Diophantine equations

$$x^2 - kxy + ky^2 + y = 0, \tag{1.5}$$

$$x^2 - kxy + ky^2 + 2y = 0, \tag{1.6}$$

$$x^2 - kxy + ky^2 + 4y = 0, \tag{1.7}$$

and

$$x^2 - kxy + ky^2 + 8y = 0 \tag{1.8}$$

have infinitely many positive integer solutions x and y, where k is a positive integer. In section 2, we give two lemmas and a theorem that will be very useful in the proof of the main theorems, and then in section 3, we give the proofs of the main theorems.

2 Preliminaries

Now we give the following lemma without proof, which will be needed in the proof of the main theorems. For the proof of it, one can see [8].

Lemma 2.1. *Let $d > 2$. If $u_0 + v_0\sqrt{d}$ is the fundamental solution of the equation $u^2 - dv^2 = \pm 2$, then $(u_0^2 + dv_0^2)/2 + u_0 v_0\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = 1$.*

The following lemma is given in [9], [10], and [11].

Lemma 2.2. *The equation $u^2 - (k^2 - 4)v^2 = -4$ has positive integer solutions u and v if and only if $k = 3$.*

Since the proof of the following theorem is given in [1], we omit its proof.

Theorem 2.3. *Let d be a positive integer which is not a perfect square. If x_1 and y_1 are natural numbers satisfying the inequality*

$$x_1 > \frac{y_1^2}{2} - 1$$

and if $\alpha = x_1 + y_1\sqrt{d}$ is a solution of the equation $x^2 - dy^2 = 1$, then α is the fundamental solution of this equation.

3 Main Theorems

Theorem 3.1. *The equation $x^2 - kxy + ky^2 + y = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5$.*

Proof. Assume that $x^2 - kxy + ky^2 + y = 0$ for some positive integers x and y. Then it follows that $y|x^2$ and thus $x^2 = yz$ for some positive integer z. A simple computation shows that $\gcd(y,z) = 1$. Then $y = a^2$ and $z = b^2$ for some positive integers a and b with $\gcd(a,b) = 1$. Thus it follows that $x = ab$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + y = 0$, we obtain

$$b^2 - kab + ka^2 + 1 = 0. \tag{3.1}$$

Next, multiplying both sides by 4 and rewriting the previous equation, we get

$$(2b - ka)^2 - \left((k-2)^2 - 4\right)a^2 = -4.$$

From Lemma 2.2, it follows that $k - 2 = 3$ and therefore $k = 5$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *The equation $x^2 - kxy + ky^2 + 2y = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5, 6$.*

Proof. Assume that $x^2 - kxy + ky^2 + 2y = 0$ for some positive integers x and y. Then it follows that $y|x^2$ and thus $x^2 = yz$ for some positive integer z. Let $d = \gcd(y,z)$. Then $y = da^2$ and

$z = db^2$ for some positive integers a and b with $\gcd(a,b) = 1$. Thus it follows that $x = dab$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 2y = 0$, we obtain

$$db^2 - kdab + kda^2 + 2 = 0, \tag{3.2}$$

which implies that $d|2$. Therefore $d = 1$ or $d = 2$. From now on we divide the proof into two cases.

Case 1 : $d = 2$. Then Eq.(3.2) turns into Eq.(3.1) and therefore it follows that $k = 5$.

Case 2 : $d = 1$. Then Eq.(3.2) becomes

$$b^2 - kab + ka^2 = -2. \tag{3.3}$$

A simple computation shows that k is even. Completing the square gives $(b-(k/2)a)^2 - \big((k/2-1)^2 - 1\big)a^2 = -2$. Let $t = k/2 - 1$. Then it follows that $(b-(k/2)a)^2 - (t^2-1)a^2 = -2$. Now we consider the equation

$$u^2 - (t^2-1)v^2 = -2. \tag{3.4}$$

Assume that $u_0 + v_0\sqrt{t^2-1}$ is the fundamental solution to Eq.(3.4). Then from Lemma 2.1, it follows that $(u_0^2 + (t^2-1)v_0^2)/2 + u_0v_0\sqrt{t^2-1}$ is the fundamental solution to the equation $x^2 - (t^2-1)y^2 = 1$. For $t > 1$, since $(t,1)$ is the fundamental solution to the equation $x^2-(t^2-1)y^2 = 1$ by Theorem 2.3, we obtain $(u_0^2 + (t^2-1)v_0^2)/2 = t$ and $u_0v_0 = 1$, which implies that $t = 2$ and thus $k = 6$. This completes the proof of Theorem 3.2. □

Theorem 3.3. *The equation $x^2 - kxy + ky^2 + 4y = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5, 6, 8$.*

Proof. Assume that $x^2 - kxy + ky^2 + 4y = 0$ for some positive integers x and y. Then it follows that $y|x^2$ and thus $x^2 = yz$ for some positive integer z. A simple computation shows that $\gcd(y,z) = 1$, or $\gcd(y,z) = 2$, or $\gcd(y,z) = 4$. Thus there exist positive integers a and b such that $y = a^2$, $z = b^2$, $x = ab$, or $y = 2a^2$, $z = 2b^2$, $x = 2ab$, or $y = 4a^2$, $z = 4b^2$, $x = 4ab$ with $\gcd(a,b) = 1$. Now we divide the remainder of the proof into three cases.

Case 1 : $\gcd(y,z) = 4$. Then $y = 4a^2$, $z = 4b^2$, $x = 4ab$, and $\gcd(a,b) = 1$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 4y = 0$, we obtain $b^2 - kab + ka^2 + 1 = 0$, which implies that $k = 5$.

Case 2 : $\gcd(y,z) = 2$. Then $y = 2a^2$, $z = 2b^2$, $x = 2ab$, and $\gcd(a,b) = 1$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 4y = 0$, we obtain $b^2 - kab + ka^2 + 2 = 0$, which implies that $k = 6$.

Case 3 : $\gcd(y,z) = 1$. Then $y = a^2$, $z = b^2$, $x = ab$, and $\gcd(a,b) = 1$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 4y = 0$, we obtain

$$b^2 - kab + ka^2 = -4. \tag{3.5}$$

In Eq.(3.5), if k is odd, then both a and b must be even, which is impossible since $\gcd(a,b) = 1$. So k is even integer. Then b is even and therefore there exists a positive integer c such that $b = 2c$. Also since $\gcd(a,b) = 1$, a is odd. Then it follows that $4|k$. Thus $k = 4s$ for some positive integer $s > 1$. Substituting these values of b and k into Eq.(3.5), we get $c^2 - 2sca + sa^2 = -1$, which gives

$$(c - sa)^2 - (s^2 - s)a^2 = -1. \tag{3.6}$$

Now we consider the equation

$$x^2 - (s^2 - s)y^2 = -1 \tag{3.7}$$

where $s > 1$. Assume that $x_0 + y_0\sqrt{s^2 - s}$ is the fundamental solution to Eq.(3.7). Then $x_0^2 + (s^2 - s)y_0^2 + 2x_0y_0\sqrt{s^2 - s}$ is the fundamental solution to the equation $x^2 - (s^2 - s)y^2 = 1$. For $s > 1$, since $(2s - 1, 2)$ is the fundamental solution to the equation $x^2 - (s^2 - s)y^2 = 1$ by Theorem 2.3, we obtain $x_0^2 + (s^2 - s)y_0^2 = 2s - 1$ and $2x_0y_0 = 2$. This shows that $x_0 = 1$ and $y_0 = 1$. Thus it follows that $s = 1$ or $s = 2$. But since $s > 1$, $s = 2$. As a consequence, we obtain $k = 8$. This completes the proof of Theorem 3.3. □

Theorem 3.4. *The equation $x^2 - kxy + ky^2 + 8y = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5, 6, 8, 12$.*

Proof. Assume that $x^2 - kxy + ky^2 + 8y = 0$ for some positive integers x and y. Then it follows that $y|x^2$ and thus $x^2 = yz$ for some positive integer z. A simple computation shows that $\gcd(y,z) = 1$, or $\gcd(y,z) = 2$, or $\gcd(y,z) = 4$, or $\gcd(y,z) = 8$. Thus there exist positive integers a and b such that $y = a^2$, $z = b^2$, $x = ab$, or $y = 2a^2$, $z = 2b^2$, $x = 2ab$, or $y = 4a^2$, $z = 4b^2$, $x = 4ab$, or $y = 8a^2$, $z = 8b^2$, $x = 8ab$ with $\gcd(a,b) = 1$. Now we divide the remainder of the proof into four cases. But the proof will only be interesting when $\gcd(y,z) = 1$ since the equation $x^2 - kxy + ky^2 + 8y = 0$ turns into Eq.(3.1) when $\gcd(y,z) = 8$, turns into Eq.(3.3) when $\gcd(y,z) = 4$, and finally turns into Eq.(3.5) when $\gcd(y,z) = 2$, respectively.

Case 1 : $\gcd(y,z) = 8$. Then $y = 8a^2$, $z = 8b^2$, $x = 8ab$, and $\gcd(a,b) = 1$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 8y = 0$, we obtain $b^2 - kab + ka^2 + 1 = 0$, which implies that $k = 5$.

Case 2 : $\gcd(y,z) = 4$. Then $y = 4a^2$, $z = 4b^2$, $x = 4ab$, and $\gcd(a,b) = 1$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 8y = 0$, we obtain $b^2 - kab + ka^2 + 2 = 0$, which implies that $k = 6$.

Case 3 : $\gcd(y,z) = 2$. Then $y = 2a^2$, $z = 2b^2$, $x = 2ab$, and $\gcd(a,b) = 1$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 8y = 0$, we obtain $b^2 - kab + ka^2 = -4$, which implies that $k = 8$.

Case 4 : $\gcd(y,z) = 1$. Then $y = a^2$, $z = b^2$, $x = ab$, and $\gcd(a,b) = 1$. Substituting these values of x and y to the equation $x^2 - kxy + ky^2 + 8y = 0$, we obtain

$$b^2 - kab + ka^2 = -8. \tag{3.8}$$

In Eq.(3.8), if k is odd, then both a and b must be even, which is impossible since $\gcd(a,b) = 1$. So k is even integer. Then b is even and therefore there exists a positive integer c such that $b = 2c$. Also since $\gcd(a,b) = 1$, a is odd and therefore a simple computation shows that $4|k$. Thus $k = 4s$ for some positive integer $s > 1$. Substituting these values of b and k into Eq.(3.8), we obtain $4c^2 - 8sca + 4sa^2 = -8$, which gives $c^2 - 2sca + sa^2 = -2$. Completing the square gives $(c - sa)^2 - (s^2 - s)a^2 = -2$. Now we consider the equation

$$u^2 - (s^2 - s)v^2 = -2. \tag{3.9}$$

If $s = 2$, then we get $u^2 - 2v^2 = -2$. Since $4 + 3\sqrt{2}$ is a solution of the equation $u^2 - 2v^2 = -2$, this equation has infinitely many solutions. Thus we get $k = 8$. Assume that $u_0 + v_0\sqrt{s^2 - s}$ be the fundamental solution to Eq.(3.9) and $s > 2$. Then from Lemma 2.1, it follows that

$((u_0^2+(s^2-s)v_0^2)/2, u_0v_0)$ is the fundamental solution to the equation $x^2-(s^2-s)y^2=1$. For $s>1$, since $(2s-1,2)$ is the fundamental solution to the equation $x^2-(s^2-s)y^2=1$ by Theorem 2.3, we obtain $(u_0^2+(s^2-s)v_0^2)/2=2s-1$ and $u_0v_0=2$. Solving these equations gives that $s=3$ and thus $k=12$. This completes the proof of Theorem 3.4. □

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