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# LARGE TIME ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATION

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**Abstract**

We consider the Cauchy problem for the higher-order nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u - \frac{1}{4}\partial_x^4 u = u^3,\ t > 0,\ x \in \mathbb{R},$$
$$u(0, x) = u_0(x),\ x \in \mathbb{R}. \tag{0.1}$$

The aim of the present paper is prove the global existence of solutions to (0.1) if the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$. Also we find the large time asymptotics of solutions.

## 1. Introduction

Consider the Cauchy problem for the higher-order nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - \frac{1}{4}\partial_x^4 u = u^3,\ t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x),\ x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Higher-order nonlinear Schrödinger equations have been introduced by Karpman [18] and Karpman-Shagalov [19] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (1.1) also occurs in the study of deep water wave dynamics [5], solitary waves [18], [19], vortex filaments [7], and so on. The local well-posedness of (1.1) has been widely studied; see [8], [9], [16], [24], [25], [26], [27], [28], [29] and references cited therein.

Note that the time decay rate of the nonlinear term in equation (1.1) is critical with respect to the large time asymptotic behavior. We call the nonlinearity gauge invariant if $\mathcal{N}\left(e^{i\vartheta}u\right) = e^{i\vartheta}\mathcal{N}(u)$ for all $\vartheta \in \mathbb{R}$. The nonlinearity $u^3$ of equation (1.1) is nongauge invariant. It is known that in the one dimensional case the cubic type nonlinearities are critical with respect to the large time asymptotic behavior of solutions (see [23]). The higher-order nonlinear Schrödinger equation with the gauge invariant term $|u|^2 u$ was considered previously in paper [13]. As far as we know the large time asymptotics of solutions to the Cauchy problem (1.1) was not studied previously. In the present paper we fill this gap, developing the factorization techniques originated in papers [15], [10], [11], [13], [14], [21], [22]. The proof of the main result in the present paper follows very closely the arguments in [12] and [20], but there are serious obstructions linked to the non-homogeneous symbol of the dispersive part that have to be solved by introducing some commutators in the analysis.

To state our results precisely we introduce some notations. We denote the Lebesgue space

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by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int |\phi(x)|^p\, dx\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x\in\mathbb{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{k,s} = \left\{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{k,s}} = \left\|\langle x\rangle^s \langle i\partial_x\rangle^k \phi\right\|_{\mathbf{L}^p} < \infty\right\},$$

$k, s \in \mathbb{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}$, $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{k,s} = \mathbf{H}_2^{k,s}$, $\mathbf{H}^k = \mathbf{H}^{k,0}$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter $C$. We denote by $\mathcal{F}\phi$ or $\hat{\phi}$ the Fourier transform $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx$, then the inverse Fourier transformation $\mathcal{F}^{-1}$ is given by $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi$.

We are in a position to state the main result of this paper. Denote $\Lambda(\xi) = \frac{1}{2}\xi^2 + \frac{1}{4}\xi^4$, $\Theta(\xi) = \frac{1}{2}\xi^2\left(1 + \frac{3}{2}\xi^2\right)$, $\mu(x) = \left(\frac{x}{2} + \sqrt{\frac{1}{4}x^2 + \frac{1}{27}}\right)^{\frac{1}{3}} - \frac{1}{3}\left(\frac{x}{2} + \sqrt{\frac{1}{4}x^2 + \frac{1}{27}}\right)^{-\frac{1}{3}}$.

**Theorem 1.1.** *Let the initial data* $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$ *with a norm* $\|u_0\|_{\mathbf{H}^1} + \|u_0\|_{\mathbf{H}^{0,1}} \le \varepsilon$, *where* $\varepsilon > 0$ *is sufficiently small. Also suppose that*

$$\sup_{|\xi|\le 1}\left|\arg\widehat{u_0}(\xi)\right| < \frac{\pi}{8}, \inf_{|\xi|\le 1}\left|\widehat{u_0}(\xi)\right| \ge \delta, \tag{1.2}$$

*where* $\delta = \varepsilon^{1+\gamma}$, $\gamma > 0$ *is small. Then there exists a unique solution* $u \in \mathbf{C}\left([1,\infty);\mathbf{H}^1 \cap \mathbf{H}^{0,1}\right)$ *of the Cauchy problem* (1.1). *Moreover the asymptotics*

$$u(t,x) = \frac{e^{it\Theta(\mu(\frac{x}{t}))}\left|\widehat{u_0}\left(\mu\left(\frac{x}{t}\right)\right)\right|}{\sqrt{it\Lambda''\left(\mu\left(\frac{x}{t}\right)\right)\left(1 + \left|\widehat{u_0}\left(\mu\left(\frac{x}{t}\right)\right)\right|^2 \log\frac{t}{1+t\mu^2\left(\frac{x}{t}\right)}\right)}} + O\left(t^{-\frac{1}{2}}\left(\log\frac{t}{1+t\mu^2\left(\frac{x}{t}\right)}\right)^{-\frac{3}{4}}\right) \tag{1.3}$$

*is valid for* $t \to \infty$ *uniformly with respect to* $x \in \mathbb{R}$.

REMARK 1.1. We believe that the conditions (1.2) are essential for the global existence of solutions. For example, if we change the sign of the nonlinearity in equation (1.1), then the solutions might blow up in a finite time.

We organize the rest of the paper as follows. Section 2 is devoted to the factorization formulas, $\mathbf{L}^2$ - boundedness of pseudodifferential operators and estimates for the defect operators $Q$ and $Q^*$. Then in Sections 3 and 4 we obtain apriori estimates of the solutions of the Cauchy problem (1.1) in the norms

$$\left\|\widehat{\varphi}\right\|_{\mathbf{X}_T} = \left\|\widehat{\varphi}\right\|_{\mathbf{Z}_T} + \sup_{t\in[0,T]}\frac{1}{K(t)}\left\|\widehat{\varphi}_\xi(t)\right\|_{\mathbf{L}^2}$$

and

$$\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_T} = \sup_{t\in[0,T]}\left(\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} + Q^{\frac{1}{2}}(t)\left\|\left\langle\xi\right\rangle^{-\gamma}\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty}\right),$$

where $K(t) = 1 + \varepsilon^2 t^{\frac{1}{4}} Q^{-\frac{3}{2}}(t)$, $Q(t) = 1 + \delta^2 \log(1+t)$. Finally, Section 5 is devoted to the proof of Theorem 1.1.

## 2. Preliminaries

**2.1. Factorization Techniques.** We define the free evolution group $\mathcal{U}(t) = \mathcal{F}^{-1} e^{-it\Lambda(\xi)} \mathcal{F}$, where $\Lambda(\xi) = \frac{1}{2}\xi^2 + \frac{1}{4}\xi^4$. We have

$$\mathcal{U}(t)\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\left(\frac{x}{t}\xi - \Lambda(\xi)\right)} \phi(\xi)\, d\xi.$$

Consider the stationary point $\mu(x)$ defined by the equation $\Lambda'(\eta) = x$. Since $\Lambda''(\xi) = 1 + 3\xi^2 > 0$, then $\Lambda'(\xi) = \xi + \xi^3$ is monotonous. Hence there exists a unique stationary point $\mu(x) = \left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + \frac{1}{27}}\right)^{\frac{1}{3}} - \frac{1}{3}\left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + \frac{1}{27}}\right)^{-\frac{1}{3}}$ such that $\Lambda'(\mu(x)) = x$ for all $x \in \mathbb{R}$. Then we write

$$\begin{aligned}
\mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{D}_t \sqrt{\frac{t}{2\pi}} e^{it\Theta(\mu(x))} \int_{\mathbb{R}} e^{-it(\Lambda(\xi) - \Lambda(\mu(x)) - x(\xi - \mu(x)))} \phi(\xi)\, d\xi \\
&= \mathcal{D}_t \mathcal{B} M \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \phi(\xi)\, d\xi = \mathcal{D}_t \mathcal{B} M \mathcal{Q}\phi,
\end{aligned}$$

where the dilation operator $\mathcal{D}_t\phi(x) = t^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right)$, the scalling operator $(\mathcal{B}\phi)(x) = \phi(\mu(x))$, the multiplication factor $M(t,\eta) = e^{it\Theta(\eta)}$, $\Theta(\eta) = \frac{1}{2}\eta^2\left(1 + \frac{3}{2}\eta^2\right)$, the phase function $S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi - \eta) = \frac{1}{4}\left(2 + \xi^2 + 2\eta\xi + 3\eta^2\right)(\xi - \eta)^2$ and the defect operator

$$\mathcal{Q}(t)\phi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \phi(\xi)\, d\xi.$$

Also we define the conjugate defect operator $\mathcal{Q}^*(t)$ by

$$\mathcal{Q}^*(t)\phi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it\Lambda(\xi) - it\xi x} e^{it\Theta(\mu(x))} \phi(\mu(x))\, dx = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \phi(\eta) \Lambda''(\eta)\, d\eta.$$

Thus we have the representation for the free evolution group $\mathcal{U}(t)\mathcal{F}^{-1} = \mathcal{D}_t \mathcal{B} M \mathcal{Q}$ and for the inverse evolution group $\mathcal{F}\mathcal{U}(-t) = \mathcal{Q}^* \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1}$ with $\left(\mathcal{B}^{-1}\phi\right)(\eta) = \phi(\Lambda'(\eta))$, $\mathcal{D}_t^{-1}\phi(x) = t^{\frac{1}{2}}\phi(xt)$. Define the operators $\mathcal{A}_k\phi = \frac{1}{t\Lambda''(\eta)} \overline{M}^k \partial_\eta M^k \phi$, $k = 0, 1$, such that $\mathcal{A}_1 = i\eta + \mathcal{A}_0$. We have $i\xi\mathcal{Q}^* = \mathcal{Q}^*\mathcal{A}_1$. Since $S(\xi,\eta) + k\Theta(\eta) = \Omega_{k+1}(\xi) + (1+k) S\left(\frac{\xi}{1+k}, \eta\right)$, where $\Omega_{k+1} = \Lambda(\xi) - (k+1)\Lambda\left(\frac{\xi}{k+1}\right)$ for $k \neq -1$, by the definition of the operator $\mathcal{Q}^*(t)$ we obtain

$$\begin{aligned}
\mathcal{Q}^*(t) M^k \phi &= \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{it(S(\xi,\eta) + k\Theta(\eta))} \phi(\eta) \Lambda''(\eta)\, d\eta \\
&= e^{it\Omega_{k+1}} \mathcal{D}_{k+1} \sqrt{\frac{(1+k)t}{2\pi}} \int_{\mathbb{R}} e^{i(1+k)tS(\xi,\eta)} \phi(\eta) \Lambda''(\eta)\, d\eta \\
&= e^{it\Omega_{k+1}} \mathcal{D}_{k+1} \mathcal{Q}^*((k+1)t)\phi.
\end{aligned} \tag{2.1}$$

We denote below $\Omega = \Omega_3 = \frac{1}{3}\xi^2\left(1 + \frac{13}{18}\xi^2\right)$. We define the new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t) u(t)$. Since $\mathcal{F}\mathcal{U}(-t)\mathcal{L} = i\partial_t \mathcal{F}\mathcal{U}(-t)$, $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{4}\partial_x^4$, applying the operator

$\mathcal{FU}(-t)$ to equation (1.1) and substituting $u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}$, in view of formula (2.1) with $k = 2$, we find the following factorization property

$$
\begin{aligned}
i\partial_t\widehat{\varphi} &= i\partial_t \mathcal{FU}(-t)\, u = \mathcal{FU}(-t)\,\mathcal{L}u = \mathcal{FU}(-t)\left(u^3\right) \\
&= \mathcal{FU}(-t)\left(\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}\right)^3 = t^{-1}\mathcal{Q}^*\overline{M}\mathcal{B}^{-1}\left(\mathcal{B}M\mathcal{Q}\widehat{\varphi}\right)^3 \\
&= t^{-1}\mathcal{Q}^*\overline{M}\left(M\mathcal{Q}\widehat{\varphi}\right)^3 = t^{-1}\mathcal{Q}^*M^2\psi^3 = t^{-1}e^{it\Omega}\mathcal{D}_3\mathcal{Q}^*(3t)\,\psi^3,
\end{aligned}
\tag{2.2}
$$

where we denote $\psi = \mathcal{Q}\widehat{\varphi}$. We next transform the nonlinear term $\mathcal{Q}^*M^2\psi^3$ in equation (2.2). Using the identity $e^{itS_3(\xi,\eta)} = H_1\partial_t\left(te^{itS_3(\xi,\eta)}\right)$ with $H_1(t,\xi,\eta) = \frac{1}{1+itS_3(\xi,\eta)}$, $S_3(\xi,\eta) = S(\xi,\eta) + 2\Theta(\eta)$, we get

$$
\begin{aligned}
&t^k\mathcal{Q}^*M^2\psi^3 = t^k\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS_3(\xi,\eta)}\psi^3\Lambda''(\eta)\,d\eta \\
&= t^k\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} \partial_t\left(te^{itS_3(\xi,\eta)}\right)H_1\psi^3\Lambda''(\eta)\,d\eta \\
&= \partial_t\left(t^k\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS_3(\xi,\eta)}tH_1\psi^3\Lambda''(\eta)\,d\eta\right) - t^k\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS_3(\xi,\eta)}H_2\psi^3\Lambda''(\eta)\,d\eta \\
&- 3t^{k+1}\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS_3(\xi,\eta)}H_1\psi^2\mathcal{Q}_t\widehat{\varphi}\Lambda''(\eta)\,d\eta - 3t^{k+1}\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS_3(\xi,\eta)}H_1\Lambda''(\eta)\,\psi^2\mathcal{Q}\widehat{\varphi}_t d\eta,
\end{aligned}
$$

where $H_2 = \frac{1+2k}{2}H_1 + t\partial_t H_1$. Therefore we find

$$
\begin{aligned}
t^k\mathcal{Q}^*M^2\psi^3 &= \partial_t\left(t^{k+1}\mathcal{Q}^*M^2H_1\psi^3\right) - t^k\mathcal{Q}^*M^2H_2\psi^3 \\
&- 3t^{k+1}\mathcal{Q}^*M^2H_1\psi^2\mathcal{Q}_t\widehat{\varphi} - 3t^{k+1}\mathcal{Q}^*M^2H_1\psi^2\mathcal{Q}\widehat{\varphi}_t.
\end{aligned}
$$

By equation (2.2) we have $\mathcal{Q}t\widehat{\varphi}_t = -i\mathcal{Q}\mathcal{Q}^*\left(M^2\psi^3\right) = -iM^2\psi^3$, then we obtain for the last summand in the above formula $\mathcal{Q}^*M^2H_1\psi^2\mathcal{Q}t\widehat{\varphi}_t = -i\mathcal{Q}^*M^4H_1\psi^5$. Thus we have

$$
t^k\mathcal{Q}^*M^2\psi^3 = \partial_t\left(t^k\mathcal{Q}^*M^2tH_1\psi^3\right) - t^k\mathcal{Q}^*M^2H_2\psi^3 - 3t^{k+1}\mathcal{Q}^*M^2H_1\psi^2\mathcal{Q}_t\widehat{\varphi} + 3it^k\mathcal{Q}^*M^4H_1\psi^5.
$$

Next we transform the derivative $\mathcal{Q}_t\widehat{\varphi}$. Denote $\psi_j = \mathcal{Q}\xi^j\widehat{\varphi}$.Using the identities $\mathcal{A}_1 = i\eta + \mathcal{A}_0$, $\mathcal{A}_k\eta = \eta\mathcal{A}_k + \frac{1}{t\Lambda''(\eta)}$, $\mathcal{A}_1\psi_j = i\psi_{j+1}$, we get $\mathcal{A}_1^2 - 2(i\eta)\mathcal{A}_1 + (i\eta)^2 = \mathcal{A}_1\mathcal{A}_0 - i\eta\mathcal{A}_0 + \frac{i}{t\Lambda''(\eta)}$ and $\mathcal{A}_1^4 - 4(i\eta)^3\mathcal{A}_1 + 3(i\eta)^4 = \mathcal{A}_1\mathcal{A}_0\mathcal{A}_1^2 + i\eta\mathcal{A}_0\mathcal{A}_1^2 - \eta^2\mathcal{A}_0\mathcal{A}_1 + 3i\eta^3\mathcal{A}_0 + \frac{i}{t\Lambda''(\eta)}\mathcal{A}_1^2$ . Then in view of equality $S(\xi,\eta) = \frac{1}{4}\left(2 + \xi^2 + 2\eta\xi + 3\eta^2\right)(\xi - \eta)^2$ , we find

$$
\begin{aligned}
\mathcal{Q}_t\widehat{\varphi} &= \frac{1}{2t}\mathcal{Q}\widehat{\varphi} - i\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}S(\xi,\eta)\,\widehat{\varphi}(\xi)\,d\xi \\
&= \frac{1}{2t}\psi - \frac{i}{2}\left(\psi_2 - 2\eta\psi_1 + \eta^2\psi\right) - \frac{i}{4}\left(\psi_4 - 4\eta^3\psi_1 + 3\eta^4\psi\right) \\
&= \frac{1}{2t}\mathcal{Q}\widehat{\varphi} + \frac{i}{2}\left(\mathcal{A}_1^2 - 2(i\eta)\mathcal{A}_1 + (i\eta)^2\right)\psi - \frac{i}{4}\left(\mathcal{A}_1^4 - 4(i\eta)^3\mathcal{A}_1 + 3(i\eta)^4\right)\psi \\
&= \frac{i}{4}\mathcal{A}_1\left(2\mathcal{A}_0\psi - \mathcal{A}_0\mathcal{A}_1^2\psi\right) + \frac{\eta}{4}\left(2\mathcal{A}_0 + \mathcal{A}_0\mathcal{A}_1^2 + i\eta\mathcal{A}_0\mathcal{A}_1 + 3\eta^2\mathcal{A}_0\right)\psi \\
&+ \frac{1}{2t\Lambda''(\eta)}\left(\Lambda''(\eta) - 1 + \frac{1}{2}\mathcal{A}_1^2\right)\psi
\end{aligned}
$$

$$= \frac{i}{4}\mathcal{A}_1\left(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2\right) + \frac{\eta}{4}\left(2\mathcal{A}_0\psi - \mathcal{A}_0\psi_2 - \eta\mathcal{A}_0\psi_1 + 3\eta^2\mathcal{A}_0\psi\right)$$
$$+ \frac{1}{2t\Lambda''(\eta)}\left(\Lambda''(\eta) - 1 + \frac{1}{2}\mathcal{A}_1^2\right)\psi.$$

Therefore we obtain

$$-3t^{k+1}Q^*M^2H_1\psi^2 Q_t\widehat{\varphi} = -\frac{3i}{4}t^{k+1}Q^*M^2H_1\psi^2\mathcal{A}_1\left(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2\right)$$
$$-\frac{3}{4}t^{k+1}Q^*M^2H_1\eta\psi^2\left(2\mathcal{A}_0\psi - \mathcal{A}_0\psi_2 - \eta\mathcal{A}_0\psi_1 + 3\eta^2\mathcal{A}_0\psi\right)$$
$$-\frac{3}{2}t^kQ^*M^2H_1\frac{1}{\Lambda''(\eta)}\psi^2\left(\left(\Lambda''(\eta) - 1\right)\psi - \frac{1}{2}\psi_2\right).$$

In the first summand on the right-hand side of the above equality we use the identity

$$Q^*M^2H_1\psi^2\mathcal{A}_1\phi = i\xi Q^*M^2H_1\psi^2\phi - 2Q^*M^2H_1\psi\psi_1\phi - Q^*M^2\left(\mathcal{A}_0H_1\right)\psi^2\phi$$

with $\phi = 2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2$, to get

$$t^kQ^*M^2\psi^3 = \partial_t\left(t^kQ^*M^2tH_1\psi^3\right) - t^kQ^*M^2H_2\psi^3 - 3t^{k+1}Q^*M^2H_1\psi^2Q_t\widehat{\varphi} + 3it^kQ^*M^4H_1\psi^5.$$

Define the kernels $H_3 = H_2 + \frac{3}{2}H_1 - \frac{3}{2\Lambda''(\eta)}H_1$, $H_4 = \frac{3}{4}\left(\xi H_1 + i\mathcal{A}_0H_1\right)$, $H_5 = \frac{3}{4\Lambda''(\eta)}H_1$. Thus we obtain the representation for the nonlinear term $t^kQ^*M^2\psi^3$ in equation (2.2) similar to the Shatah normal forms

(2.3)

$$t^kQ^*M^2\psi^3 = \partial_t\left(t^{k+1}Q^*M^2H_1\psi^3\right) - t^kQ^*M^2H_3\psi^3 + t^kQ^*M^2H_5\psi^2\psi_2 + 3it^kQ^*M^4H_1\psi^5$$
$$+ \frac{3i}{2}t^{k+1}Q^*M^2H_1\psi\psi_1\left(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2\right) + t^{k+1}Q^*M^2H_4\psi^2\left(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2\right)$$
$$-\frac{3}{4}t^{k+1}Q^*M^2\eta H_1\psi^2\left(\left(2 + 3\eta^2\right)\mathcal{A}_0\psi - \eta\mathcal{A}_0\psi_1 - \mathcal{A}_0\psi_2\right)$$

for $k = 0, -1$. Finally we note that $S(\xi,\eta) + 2\Theta(\eta) = \Omega(\xi) + 3S\left(\frac{\xi}{3},\eta\right)$, so we have the estimate

$$S_3(\xi,\eta) = \frac{3}{4}\Omega(\xi) + \frac{1}{2}\Theta(\eta) + \frac{3}{4}\left(S(\xi,\eta) + S\left(\frac{\xi}{3},\eta\right)\right) \geq \frac{1}{8}\xi^2\langle\xi\rangle^2 + \frac{1}{8}\eta^2\langle\eta\rangle^2.$$

**2.2. Boundedness of pseudodifferential operators.** There are many papers (see, e.g. [1], [3], [4]) on the $\mathbf{L}^2$ - estimates of pseudodifferential operator $\mathbf{a}(x,D)\phi \equiv \int_{\mathbb{R}} e^{ix\xi}\mathbf{a}(x,\xi)\widehat{\phi}(\xi)\,d\xi$. Below we will use the following result on the $\mathbf{L}^2$ - boundedness of $\mathbf{a}(x,D)$ (see [17]).

**Lemma 2.1.** *Let the symbol* $\mathbf{a}(x,\xi)$ *be such that* $\sup_{x,\xi\in\mathbb{R}}\left|\partial_x^m\partial_\xi^n\mathbf{a}(x,\xi)\right| \leq C$ *for* $m,n = 0,1$. *Then* $\|\mathbf{a}(x,D)\phi\|_{\mathbf{L}^2_x} \leq C\|\phi\|_{\mathbf{L}^2}$.

**2.3. Estimates for defect operator.** Define the kernel $A_j(t,\eta) = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\xi^j d\xi$. To compute the asymptotics of the kernel $A_j(t,\eta)$ for large time we apply the stationary phase method (see [6], p. 110)

$$\int_{\mathbb{R}} e^{itg(y)}f(y)\,dy = e^{izg(y_0)}f(y_0)\sqrt{\frac{2\pi}{t\left|g''(y_0)\right|}}e^{i\frac{\pi}{4}\operatorname{sgn}g''(y_0)} + O\left(t^{-\frac{3}{2}}\right) \tag{2.4}$$

for $t \to \infty$, where the stationary point $y_0$ is defined by the equation $g'(y_0) = 0$. By virtue of formula (2.4) with $g(y) = -S(y,\eta)$, $f(y) = y^j$, $y_0 = \eta$, we get $A_j(t,\eta) = \frac{\eta^j}{\sqrt{i\Lambda''(\eta)}} + O\left(t^{-1}\right)$ for $t \to \infty$. Since $\Lambda''(\eta) = O\left(\langle\eta\rangle^2\right)$ we also have the estimate $\left|A_j(t,\eta)\right| \le \langle\eta\rangle^{j-1}$. In the next lemma we find the asymptotics for $Q\xi^j\phi$.

**Lemma 2.2.** *The estimate* $\left\|\langle\eta\rangle^{\frac{3}{2}-j}\left(Q\xi^j\phi - A_j\phi\right)\right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{4}}\left\|\phi_\xi\right\|_{\mathbf{L}^2}$ *is valid for all* $t \ge 1$, $j = 0,1,2$.

Proof. We integrate by parts via the identity $e^{-itS(\xi,\eta)} = W\partial_\xi\left((\xi-\eta)e^{-itS(\xi,\eta)}\right)$ with $W = \left(1 - it(\xi-\eta)\partial_\xi S(\xi,\eta)\right)^{-1}$, then we get

$$Q\xi^j\phi - A_j\phi = Ct^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\left(\phi(\xi) - \phi(\eta)\right)\xi^j d\xi$$

$$= Ct^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\left(\phi(\xi) - \phi(\eta)\right)(\xi-\eta)\partial_\xi\left(\xi^jW\right)d\xi + Ct^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}(\xi-\eta)W\xi^j\phi_\xi(\xi)\,d\xi.$$

Since $\left|S_\xi(\xi,\eta)\right| \ge C\left|\xi-\eta\right|\left(\langle\xi\rangle^2 + \langle\eta\rangle^2\right)$, we find $|W| \le \frac{C}{1+t(\xi-\eta)^2\left(\langle\xi\rangle^2+\langle\eta\rangle^2\right)}$ and $\left|(\xi-\eta)\partial_\xi\left(\xi^jW\right)\right| \le \frac{C|\xi|^j}{1+t(\xi-\eta)^2\left(\langle\xi\rangle^2+\langle\eta\rangle^2\right)}$. Thus by the Cauchy-Schwarz and Hardy inequalities we obtain

$$\left|Q\xi^j\phi - A_j\phi\right| \le Ct^{\frac{1}{2}}\int_{\mathbb{R}}\left(\frac{|\phi(\xi) - \phi(\eta)|}{|\xi-\eta|} + \left|\phi_\xi(\xi)\right|\right)\frac{|\xi-\eta|\,|\xi|^j\,d\xi}{1 + t(\xi-\eta)^2\left(\langle\xi\rangle^2 + \langle\eta\rangle^2\right)} \le C\left\|\phi_\xi\right\|_{\mathbf{L}^2}\sqrt{tI},$$

where $I = \int_{\mathbb{R}}\frac{(\xi-\eta)^2|\xi|^{2j}d\xi}{\left(1+t(\xi-\eta)^2\left(\langle\xi\rangle^2+\langle\eta\rangle^2\right)\right)^2}$. Since $I \le Ct^{-\frac{3}{2}}\langle\eta\rangle^{2j-3}$ we get $\left|Q\xi^j\phi - A_j\phi\right| \le Ct^{-\frac{1}{4}}\langle\eta\rangle^{j-\frac{3}{2}}\left\|\phi_\xi\right\|_{\mathbf{L}^2}$. Lemma 2.2 is proved. □

In the next lemma we estimate derivative $\partial_\eta Q$.

**Lemma 2.3.** *The following estimate is true*

$$\left\|\langle\eta\rangle^{-1}\eta^l\partial_\eta Q\xi^j\phi\right\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} + C\left|\phi(0)\right|$$

*for all* $t \ge 1$, *if* $l, j \in \mathbb{Z}_+$, $l + j \le 2$.

Proof. Integrating by parts we obtain

$$\eta^l\partial_\eta Q\xi^j\phi = C\Lambda''(\eta)\,t^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\partial_\xi\left(\frac{(\eta-\xi)\,\xi^j\eta^l}{\partial_\xi S(\xi,\eta)}\phi(\xi)\right)d\xi = I_1 + I_2 + I_3,$$

where

$$I_k = C\Lambda''(\eta)\,t^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_k(\eta,\xi)\,z_k(\xi)\,d\xi,\ k = 1,2,3$$

with $z_1(\xi) = \phi_\xi(\xi)$, $z_2(\xi) = \frac{\phi(\xi)-\phi(0)}{\xi}$, $z_3(\xi) = \phi(0)\langle\xi\rangle^{-1}$,

$$q_k(\eta,\xi) = \left(\xi\partial_\xi\right)^{k-1}\left(\frac{\xi^j\eta^l}{1 + \xi^2 + \eta\xi + \eta^2}\right),\ k = 1,2,$$

and $q_3(\eta,\xi) = \langle\xi\rangle\partial_\xi\left(\frac{\xi^j\eta^l}{1+\xi^2+\eta\xi+\eta^2}\right)$, since $\partial_\xi S(\xi,\eta) = (\xi-\eta)\left(1 + \xi^2 + \eta\xi + \eta^2\right)$, $\partial_\eta S(\xi,\eta) = \Lambda''(\eta)(\eta-\xi)$. Next we change $\eta = \mu(x)$, then we get

$$I_k = \Lambda''(\eta)\,\overline{\mathcal{M}}\mathcal{B}^{-1}t^{\frac{1}{2}}\int_{\mathbb{R}} e^{itx\xi}q_k(\mu(x),\xi)\,e^{-it\Lambda(\xi)}z_k(\xi)\,d\xi.$$

After that we change the variable of integration $\xi = t^{-\frac{1}{2}}\xi'$, then

$$I_k = \Lambda''(\eta)\,\overline{\mathcal{M}}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ix\xi}q_k\left(\mu\left(xt^{-\frac{1}{2}}\right),\xi t^{-\frac{1}{2}}\right)\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_k d\xi.$$

where $\mathcal{D}_{t^{\frac{1}{2}}}\phi(x) = t^{-\frac{1}{4}}\phi\left(\frac{x}{\sqrt{t}}\right)$, $\mathcal{D}^{-1}_{t^{\frac{1}{2}}}\phi(x) = t^{\frac{1}{4}}\phi\left(x\sqrt{t}\right)$. Define the pseudodifferential operators $\mathbf{a}_k(t,x,D)\phi \equiv \int_{\mathbb{R}} e^{ix\xi}\mathbf{a}_k(t,x,\xi)\widehat{\phi}(\xi)\,d\xi$ with the symbols $\mathbf{a}_k(t,x,\xi) = q_k\left(\mu\left(xt^{-\frac{1}{2}}\right),\xi t^{-\frac{1}{2}}\right)$, $k = 1,2,3$. Then we get $I_k = \Lambda''(\eta)\,\overline{\mathcal{M}}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{\frac{1}{2}}}\mathbf{a}_k(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_k$. Let us prove the $\mathbf{L}^2$ - boundedness of the pseudodifferential operators $\mathbf{a}_k(t,x,D)$, $k = 1,2,3$. Since $\mu(x) = O\left(\{x\}\langle x\rangle^{\frac{1}{3}}\right)$ with $\{x\} = \frac{|x|}{\langle x\rangle}$, we obtain

$$\mathbf{a}_k(t,x,\xi) = -\left(\xi\partial_\xi\right)^{k-1}\left.\left(\frac{\left(\xi t^{-\frac{1}{2}}\right)^j\eta^l}{1+\xi^2t^{-1}+\eta\xi t^{-\frac{1}{2}}+\eta^2}\right)\right|_{\eta=\mu\left(xt^{-\frac{1}{2}}\right)}$$

for $k = 1,2$. Since $\mu'(x) = \frac{1}{\Lambda''(\mu(x))}$, we get

$$\left|\partial_x^m\partial_\xi^n\mathbf{a}_k(t,x,\xi)\right| = O\left(\frac{t^{-\frac{m}{2}}}{(\Lambda''(\eta))^m}\partial_\eta^m\partial_\xi^n\left(\xi\partial_\xi\right)^{k-1}\left.\left(\frac{\left(\xi t^{-\frac{1}{2}}\right)^j\eta^l}{1+\xi^2t^{-1}+\eta\xi t^{-\frac{1}{2}}+\eta^2}\right)\right|_{\eta=\mu\left(xt^{-\frac{1}{2}}\right)}\right) \le C$$

for all $x,\xi\in\mathbb{R}$, $t\ge 1$, $m,n = 0,1$, $k = 1,2$, if $l,j\in\mathbb{Z}_+$, $l+j\le 2$. Therefore by Lemma 2.1 we find $\|\mathbf{a}_k(t,x,D)\phi\|_{\mathbf{L}^2_x} \le C\|\phi\|_{\mathbf{L}^2}$. Then we get

$$\begin{aligned}\left\|\langle\eta\rangle^{-1}I_k\right\|_{\mathbf{L}^2} &\le C\left\|\left|\Lambda''\right|^{\frac{1}{2}}\overline{\mathcal{M}}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{\frac{1}{2}}}\mathbf{a}_k(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_k\right\|_{\mathbf{L}^2_\eta}\\ &\le \left\|\mathbf{a}_k(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_k\right\|_{\mathbf{L}^2_x} \le C\left\|\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_k\right\|_{\mathbf{L}^2} = C\|z_k\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2},\end{aligned}$$

since by the Hardy inequality $\|z_2\|_{\mathbf{L}^2} = \left\|\frac{\phi(\xi)-\phi(0)}{\xi}\right\|_{\mathbf{L}^2} \le \left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}$. Since $\mu'(x) = \frac{1}{\Lambda''(\mu(x))}$, we get

$$\left|\partial_x^m\partial_\xi^n\mathbf{a}_3(t,x,\xi)\right| = O\left(\frac{t^{-\frac{m}{2}}}{(\Lambda''(\eta))^m}\partial_\eta^m\partial_\xi^n\left\langle\xi t^{-\frac{1}{2}}\right\rangle\partial_\xi\left.\left(\frac{\left(\xi t^{-\frac{1}{2}}\right)^j\eta^l}{1+\xi^2t^{-1}+\eta\xi t^{-\frac{1}{2}}+\eta^2}\right)\right|_{\eta=\mu\left(xt^{-\frac{1}{2}}\right)}\right) \le C$$

for all $x,\xi\in\mathbb{R}$, $t\ge 1$, $m,n = 0,1$, if $l,j\in\mathbb{Z}_+$, $l+j\le 2$. Therefore by Lemma 2.1 we find $\|\mathbf{a}_3(t,x,D)\phi\|_{\mathbf{L}^2_x} \le C\|\phi\|_{\mathbf{L}^2}$. Then we get

$$\begin{aligned}\left\|\langle\eta\rangle^{-1}I_3\right\|_{\mathbf{L}^2} &\le C\left\|\left|\Lambda''\right|^{\frac{1}{2}}\overline{\mathcal{M}}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{\frac{1}{2}}}\mathbf{a}_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_3\right\|_{\mathbf{L}^2_\eta}\\ &\le \left\|\mathbf{a}_k(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_3\right\|_{\mathbf{L}^2_x} \le C\left\|\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}e^{-it\Lambda}z_3\right\|_{\mathbf{L}^2}\\ &= C\|z_3\|_{\mathbf{L}^2} \le C|\phi(0)|.\end{aligned}$$

Lemma 2.3 is proved. □

**2.4. Estimates for conjugate defect operator.** In the next lemma we obtain the $\mathbf{L}^{\infty}$ estimate for $Q^{*}H$. Denote $\widetilde{\xi}=\xi\sqrt{t}, \widetilde{\eta}=\eta\sqrt{t}$.

**Lemma 2.4.** *Let the kernel $H$ satisfy $|H(t,\xi,\eta)|\leq C\left(\left\langle\widetilde{\xi}\right\rangle+\langle\widetilde{\eta}\rangle\right)^{-j}$ for all $\xi,\eta\in\mathbb{R}$, $t\geq1$, where $j\geq0$. Then the estimate $\left\|\left\langle\widetilde{\xi}\right\rangle^{j-k}Q^{*}H\phi\right\|_{\mathbf{L}^{\infty}}\leq Ct^{\frac{1}{2}}\left\|\langle\eta\rangle^{2}\langle\widetilde{\eta}\rangle^{-k}\phi\right\|_{\mathbf{L}^{1}}$ is true for all $t\geq1$, if $0\leq k\leq j$.*

Proof. We have

$$\left|\left\langle\widetilde{\xi}\right\rangle^{j-k}Q^{*}H\phi\right|\leq Ct^{\frac{1}{2}}\left|\left\langle\widetilde{\xi}\right\rangle^{j-k}\int_{\mathbb{R}}e^{itS(\xi,\eta)}H(t,\xi,\eta)\phi(\eta)\Lambda''(\eta)\,d\eta\right|$$

$$\leq Ct^{\frac{1}{2}}\int_{\mathbb{R}}\frac{\left\langle\widetilde{\xi}\right\rangle^{j-k}\langle\widetilde{\eta}\rangle^{k}}{\left(\left\langle\widetilde{\xi}\right\rangle+\langle\widetilde{\eta}\rangle\right)^{a}}\langle\widetilde{\eta}\rangle^{-k}|\phi(\eta)|\Lambda''(\eta)\,d\eta$$

$$\leq Ct^{\frac{1}{2}}\int_{\mathbb{R}}|\phi(\eta)|\langle\widetilde{\eta}\rangle^{-k}\Lambda''(\eta)\,d\eta=Ct^{\frac{1}{2}}\left\|\langle\eta\rangle^{2}\langle\widetilde{\eta}\rangle^{-k}\phi\right\|_{\mathbf{L}^{1}}.$$

Lemma 2.4 is proved. □

Next we obtain the large time asymptotics of the operator $Q^{*}H$. Define the conjugate kernel $A_{H}^{*}(t,\xi)=\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}}e^{itS(\xi,\eta)}H(t,\xi,\eta)\Lambda''(\eta)\,d\eta$.

**Lemma 2.5.** *Let the kernel $H$ satisfy the estimate $|H(t,\xi,\eta)|\leq C\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{-2}$ for all $\xi,\eta\in\mathbb{R}$, $t\geq1$. Then the estimate is true*

$$\left\|Q^{*}H\phi-A_{H}^{*}\phi\right\|_{\mathbf{L}^{\infty}}\leq Ct^{-\frac{1}{4}}\left\|\langle\widetilde{\eta}\rangle^{-\alpha}\phi_{\eta}\right\|_{\mathbf{L}^{2}}$$

*for all $t\geq1$, where $\alpha\in\left(0,\frac{1}{2}\right)$.*

Proof. We have

$$\left(\left\langle\widetilde{\xi}\right\rangle+\langle\widetilde{\eta}\rangle\right)^{-\alpha}|\phi(\eta)-\phi(\xi)|=\left(\left\langle\widetilde{\xi}\right\rangle+\langle\widetilde{\eta}\rangle\right)^{-\alpha}\left|\int_{\eta}^{\xi}\phi_{z}(z)\,dz\right|$$

$$\leq C\int_{\eta}^{\xi}\langle\widetilde{z}\rangle^{-\alpha}|\phi_{z}(z)|\,dz\leq C\sqrt{|\eta-\xi|}\left\|\langle\widetilde{\eta}\rangle^{-\alpha}\phi_{\eta}\right\|_{\mathbf{L}^{2}}.$$

Then we find

$$\left|Q^{*}H\phi-A_{H}^{*}\phi\right|=Ct^{\frac{1}{2}}\left|\int_{\mathbb{R}}e^{itS(\xi,\eta)}(\phi(\eta)-\phi(\xi))H\Lambda''d\eta\right|$$

$$\leq Ct^{\frac{1}{2}}\int_{\mathbb{R}}|\phi(\eta)-\phi(\xi)||H(t,\xi,\eta)|\langle\eta\rangle^{2}\,d\eta\leq Ct^{\frac{1}{2}}\left\|\langle\widetilde{\eta}\rangle^{-\alpha}\phi_{\eta}\right\|_{\mathbf{L}^{2}}I,$$

where $I=\int_{\mathbb{R}}\frac{\sqrt{|\eta-\xi|}\left(\left\langle\widetilde{\xi}\right\rangle+\langle\widetilde{\eta}\rangle\right)^{\alpha}\langle\eta\rangle^{2}d\eta}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}$. We have

$$I\leq Ct^{-\frac{3}{4}}\int_{\mathbb{R}}\frac{\left|\widetilde{\eta}-\widetilde{\xi}\right|^{\frac{1}{2}}\left(\left\langle\widetilde{\xi}\right\rangle+\langle\widetilde{\eta}\rangle\right)^{\alpha}\langle\eta\rangle^{2}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}$$

$$\leq Ct^{-\frac{3}{4}}\left(\int_{|\widetilde{\eta}|\leq|\widetilde{\xi}|}+\int_{|\widetilde{\eta}|\geq|\widetilde{\xi}|}\right)\frac{\left(\left\langle\widetilde{\xi}\right\rangle+\langle\widetilde{\eta}\rangle\right)^{\alpha+\frac{1}{2}}\langle\eta\rangle^{2}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}$$

$$\leq Ct^{-\frac{3}{4}}\left(\int_{|\widetilde{\eta}|\leq|\widetilde{\xi}|}\frac{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle\right)^{\alpha+\frac{1}{2}}\langle\eta\rangle^{\frac{3}{2}-\alpha}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}+\int_{|\widetilde{\eta}|\geq|\widetilde{\xi}|}\frac{\left(\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{\alpha+\frac{1}{2}}\langle\eta\rangle^{\frac{3}{2}-\alpha}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}\right)$$

$$\leq Ct^{-\frac{3}{4}}\int_{\mathbb{R}}\langle\widetilde{\eta}\rangle^{\alpha-\frac{3}{2}}\,d\widetilde{\eta}\leq Ct^{-\frac{3}{4}}.$$

Therefore we get $\left|Q^{*}H\phi-A_{H}^{*}\phi\right|\leq Ct^{-\frac{1}{4}}\left\|\langle\widetilde{\eta}\rangle^{-\alpha}\phi_{\eta}\right\|_{\mathbf{L}^{2}}$. Lemma 2.5 is proved. □

Define the conjugate kernel $A_{H_3}^{*}(t,\xi)=\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}}e^{itS_3(\xi,\eta)}H_3(t,\xi,\eta)\Lambda''(\eta)\,d\eta$.

**Lemma 2.6.** *The estimates are true*

$$\left|A_{H_3}^{*}(t,\xi)\right|\leq C\left\langle\widetilde{\xi}\right\rangle^{-\alpha},\quad\left|A_{H_3}^{*}(t,\xi)-A_{H_3}^{*}(t,0)\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}}\right|\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}}$$

*for all* $\xi\in\mathbb{R}$, $t\geq1$, *where* $\alpha\in\left(0,\frac{1}{2}\right)$. *Also the asymptotics is valid*

$$A_{H_3}^{*}(t,0)=-\sqrt{i}\Phi+O\left(t^{-1}\right)$$

*for all* $t\geq1$, *where* $\Phi=\frac{1}{\sqrt{\pi i}}\int_{0}^{\infty}e^{3iy^{2}}\frac{\left(1+9iy^{2}\right)dy}{\left(1+3iy^{2}\right)^{2}}\approx0.577$.

Proof. Since $tS_3(\xi,\eta)\geq\frac{1}{8}\left(\widetilde{\xi}^{2}\langle\xi\rangle^{2}+\widetilde{\eta}^{2}\langle\eta\rangle^{2}\right)$, then the kernel $H_3=H_1^{2}-\frac{3}{2\Lambda''(\eta)}H_1$, $H_1(t,\xi,\eta)=\frac{1}{1+itS_3(\xi,\eta)}$, satisfies the estimate $|H_3(t,\xi,\eta)|\leq C\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{-2}$ for all $\xi,\eta\in\mathbb{R}$, $t\geq1$. Hence

$$\left\langle\widetilde{\xi}\right\rangle^{\alpha}\left|A_{H}^{*}(t,\xi)\right|=\left\langle\widetilde{\xi}\right\rangle^{\alpha}\left|\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}}e^{itS_3(\xi,\eta)}H(t,\xi,\eta)\Lambda''(\eta)\,d\eta\right|$$

$$\leq C\left(\int_{|\widetilde{\eta}|\leq|\widetilde{\xi}|}+\int_{|\widetilde{\eta}|\geq|\widetilde{\xi}|}\right)\frac{\left\langle\widetilde{\xi}\right\rangle^{\alpha}\langle\eta\rangle^{2}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}$$

$$\leq C\left(\int_{|\widetilde{\eta}|\leq|\widetilde{\xi}|}\frac{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle\right)^{\alpha}\langle\eta\rangle^{2-\alpha}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}+\int_{|\widetilde{\eta}|\geq|\widetilde{\xi}|}\frac{\left(\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{\alpha}\langle\eta\rangle^{2-\alpha}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}\right)$$

$$\leq C\int_{\mathbb{R}}\frac{\langle\eta\rangle^{2-\alpha}\,d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2-\alpha}}\leq C\int_{\mathbb{R}}\langle\widetilde{\eta}\rangle^{\alpha-2}\,d\widetilde{\eta}\leq C.$$

Note that $S_3(\xi,\eta)-S_3(0,\eta)=\frac{1}{4}\xi\left(2\xi+\xi^{3}-4\eta-4\eta^{3}\right)$, therefore

$$\left|e^{itS_3(\xi,\eta)}-e^{itS_3(0,\eta)}\right|\leq t^{\frac{\alpha}{2}}\left|S_3(\xi,\eta)-S_3(0,\eta)\right|^{\frac{\alpha}{2}}$$

$$=Ct^{\frac{\alpha}{2}}\left|\xi\left(2\xi+\xi^{3}-4\eta-4\eta^{3}\right)\right|^{\frac{\alpha}{2}}\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^{2}+\langle\widetilde{\eta}\rangle\langle\eta\rangle^{2}\right)^{\frac{\alpha}{2}}.$$

Hence

$$t^{\frac{1}{2}}\int_{\mathbb{R}}\left|e^{itS_3(\xi,\eta)}-e^{itS_3(0,\eta)}\right|\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}H(t,\xi,\eta)\Lambda''(\eta)\,d\eta$$

$$\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\int_{\mathbb{R}}\frac{\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^{2}+\langle\widetilde{\eta}\rangle\langle\eta\rangle^{2}\right)^{\frac{\alpha}{2}}\langle\eta\rangle^{2}d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}$$
$$\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\left(\int_{|\widetilde{\eta}|\leq|\widetilde{\xi}|}\frac{\left\langle\widetilde{\xi}\right\rangle^{\alpha}\langle\xi\rangle^{\alpha}\langle\eta\rangle^{2}d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}+\int_{|\widetilde{\eta}|\geq|\widetilde{\xi}|}\frac{\langle\widetilde{\eta}\rangle^{\alpha}\langle\eta\rangle^{2+\alpha}d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2}}\right)$$
$$\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\int_{\mathbb{R}}\frac{\langle\eta\rangle^{2}d\widetilde{\eta}}{\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{2-\alpha}}\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\int_{\mathbb{R}}\langle\widetilde{\eta}\rangle^{2\alpha-2}d\widetilde{\eta}\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}.$$

Also we find

$$\begin{aligned}&|H_3(t,\xi,\eta)-H_3(t,0,\eta)|\\&\leq Ct^{\frac{\alpha}{2}}|S_3(\xi,\eta)-S_3(0,\eta)|^{\frac{\alpha}{2}}\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{-\sigma}(\langle\widetilde{\eta}\rangle\langle\eta\rangle)^{-2}\\&\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^{2}+\langle\widetilde{\eta}\rangle\langle\eta\rangle^{2}\right)^{\frac{\alpha}{2}}\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{-\alpha}(\langle\widetilde{\eta}\rangle\langle\eta\rangle)^{-2}\\&\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}(\langle\widetilde{\eta}\rangle\langle\eta\rangle)^{-2}\end{aligned}$$

Hence

$$\left|\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}H_3(t,\xi,\eta)-H_3(t,0,\eta)\right|\leq\left|\left(\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}-1\right)H_3(t,\xi,\eta)\right|+|H_3(t,\xi,\eta)-H_3(t,0,\eta)|$$
$$\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\left(\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle+\langle\widetilde{\eta}\rangle\langle\eta\rangle\right)^{-2}+C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}(\langle\widetilde{\eta}\rangle\langle\eta\rangle)^{-2}\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}(\langle\widetilde{\eta}\rangle\langle\eta\rangle)^{-2}.$$

Then we get

$$t^{\frac{1}{2}}\left|\int_{\mathbb{R}}\left(\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}H(t,\xi,\eta)-H(t,0,\eta)\right)\Lambda''(\eta)\,d\eta\right|\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}\int_{\mathbb{R}}\langle\widetilde{\eta}\rangle^{-2}d\widetilde{\eta}\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}.$$

Therefore we obtain

$$\begin{aligned}&\left|A_H^*(t,\xi)\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}-A_H^*(t,0)\right|\\&=Ct^{\frac{1}{2}}\left|\int_{\mathbb{R}}\left(e^{itS_3(\xi,\eta)}\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}H(t,\xi,\eta)-e^{itS_3(0,\eta)}H(t,0,\eta)\right)\Lambda''(\eta)\,d\eta\right|\\&\leq Ct^{\frac{1}{2}}\int_{\mathbb{R}}\left|e^{itS_3(\xi,\eta)}-e^{itS_3(0,\eta)}\right|\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}H(t,\xi,\eta)\,\Lambda''(\eta)\,d\eta\\&+Ct^{\frac{1}{2}}\int_{\mathbb{R}}\left(\left\langle\widetilde{\xi}\right\rangle^{\frac{\alpha}{2}}H(t,\xi,\eta)-H(t,0,\eta)\right)\Lambda''(\eta)\,d\eta\leq C\left\{\widetilde{\xi}\right\}^{\frac{\alpha}{2}}.\end{aligned}$$

Finally we consider $A_{H_3}^*(t,0)$. We have $S_3(0,\eta)=3\Theta(\eta)$,

$$H_3(t,0,\eta)=\frac{1}{(1+3it\Theta(\eta))^2}-\frac{3}{2\Lambda''(\eta)(1+3it\Theta(\eta))}.$$

Then

$$\begin{aligned}A_{H_3}^*(t,0)&=\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}}e^{itS_3(0,\eta)}H_3(t,0,\eta)\,\Lambda''(\eta)\,d\eta\\&=\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}}e^{3it\Theta(\eta)}\left(\frac{\Lambda''(\eta)}{(1+3it\Theta(\eta))^2}-\frac{3}{2(1+3it\Theta(\eta))}\right)d\eta.\end{aligned}$$

We change $\Theta(\eta) = z^2$, i.e. $\eta^2 = \frac{1}{3}\left(\sqrt{12z^2+1} - 1\right)$, $\frac{dz}{d\eta} = \frac{1}{2}\frac{1+3\eta^2}{\sqrt{\frac{3}{4}\eta^2+\frac{1}{2}}}$, then we get

$$A^*_{H_3}(t,0) = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{3itz^2}\frac{\left(\left(2\sqrt{1+12z^2} - 3\left(1+3itz^2\right)\right)\right)\sqrt{1+\sqrt{1+12z^2}}}{2\left(1+3itz^2\right)^2\sqrt{1+12z^2}}dz.$$

Next we change $z\sqrt{t} = y$, then

$$\begin{aligned} A^*_{H_1}(t,0) &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{3iy^2}\frac{\left(2\sqrt{1+\frac{12}{t}y^2} - 3\left(1+3iy^2\right)\right)\sqrt{1+\sqrt{1+\frac{12}{t}y^2}}}{2\left(1+3iy^2\right)^2\sqrt{1+\frac{12}{t}y^2}}dy \\ &= -\sqrt{i}\Phi + O\left(t^{-1}\right), \end{aligned}$$

where $\Phi = \frac{1}{\sqrt{\pi i}}\int_0^\infty e^{3iy^2}\frac{\left(1+9iy^2\right)dy}{\left(1+3iy^2\right)^2} \approx 0.577$. Lemma 2.6 is proved. □

In the next lemma we prove the $\mathbf{L}^2$ - boundedness of the operator $\mathcal{Q}^*H$ uniformly with respect to $t \geq 1$.

**Lemma 2.7.** *Let the kernel $H$ satisfy the estimates $\left|\partial_\eta^m\partial_\xi^n H(t,\xi,\eta)\right| \leq Ct^{\frac{m+n}{2}}$ for all $\xi,\eta \in \mathbb{R}$, $t \geq 1$, where $m,n = 0,1$. Then the estimate $\sup_{t\geq1}\|\mathcal{Q}^*H\phi\|_{\mathbf{L}^2} \leq C\|\langle\eta\rangle\phi\|_{\mathbf{L}^2}$ is true for all $t \geq 1$.*

Proof. We change the variable of integration $\eta = \mu(x)$, then we get

$$\begin{aligned} \mathcal{Q}^*H\phi &= Ct^{\frac{1}{2}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}H(t,\xi,\eta)\phi(\eta)\Lambda''(\eta)\,d\eta \\ &= e^{it\Lambda(\xi)}t^{\frac{1}{2}}\int_{\mathbb{R}} e^{-itx\xi}H(t,\xi,\mu(x))\,\mathcal{B}M\phi dx. \end{aligned}$$

After that we change the variable of integration $x = t^{-\frac{1}{2}}x'$, then we can see that $\mathcal{Q}^*H\phi$ is the Fourier type integral

$$\mathcal{Q}^*H\phi = e^{it\Lambda(\xi)}\mathcal{D}^{-1}_{t^{\frac{1}{2}}}\int_{\mathbb{R}} e^{-ix\xi}H\left(t,\xi t^{-\frac{1}{2}},\mu\left(xt^{-\frac{1}{2}}\right)\right)\mathcal{D}_{t^{\frac{1}{2}}}\mathcal{B}M\phi dx.$$

Define the pseudodifferential operator $\mathbf{a}(t,\xi,D)\phi \equiv \int_{\mathbb{R}} e^{-ix\xi}\mathbf{a}(t,\xi,x)\widehat{\phi}(x)\,dx$ with the symbols $\mathbf{a}(t,\xi,x) = H\left(t,\xi t^{-\frac{1}{2}},\mu\left(xt^{-\frac{1}{2}}\right)\right)$. Then we get $\mathcal{Q}^*H\phi = \mathcal{D}^{-1}_{t^{\frac{1}{2}}}\mathbf{a}(t,\xi,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}\mathcal{B}M\phi$. Let us prove the $\mathbf{L}^2$ - boundedness of the pseudodifferential operator $\mathbf{a}(t,\xi,D)$. We obtain for the symbol $\mathbf{a}(t,\xi,x) = H\left(t,\xi t^{-\frac{1}{2}},\eta\right)\Big|_{\eta=\mu\left(xt^{-\frac{1}{2}}\right)}$. Since $\mu'(x) = \frac{1}{\Lambda''(\mu(x))}$, we get

$$\left|\partial_x^m\partial_\xi^n\mathbf{a}(t,\xi,x)\right| = O\left(\frac{t^{-\frac{m}{2}}}{\left(\Lambda''(\eta)\right)^m}\partial_\eta^m\partial_\xi^n H\left(t,\xi t^{-\frac{1}{2}},\eta\right)\Big|_{\eta=\mu\left(xt^{-\frac{1}{2}}\right)}\right) \leq C$$

for all $x,\xi \in \mathbb{R}$, $t \geq 1$, $m,n = 0,1$. Therefore by Lemma 2.1 we find $\|\mathbf{a}(t,\xi,D)\phi\|_{\mathbf{L}^2_\xi} \leq C\|\phi\|_{\mathbf{L}^2}$. Then we get

$$\|\mathcal{Q}^*H\phi\|_{\mathbf{L}^2} \leq C\left\|\mathcal{D}^{-1}_{t^{\frac{1}{2}}}\mathbf{a}(t,\xi,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}\mathcal{B}M\phi\right\|_{\mathbf{L}^2_\xi}$$

$$\leq \left\| \mathbf{a}\left(t,\xi,D\right)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}\mathcal{B}M\phi \right\|_{\mathbf{L}^2_\xi} \leq C\left\| \mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{2}}}\mathcal{B}M\phi \right\|_{\mathbf{L}^2}$$
$$= C\left\| \mathcal{D}_{t^{\frac{1}{2}}}\mathcal{B}M\phi \right\|_{\mathbf{L}^2} = C\left\|\mathcal{B}M\phi\right\|_{\mathbf{L}^2} = C\left\| \sqrt{\Lambda''}\phi \right\|_{\mathbf{L}^2} \leq C\left\|\langle\eta\rangle\phi\right\|_{\mathbf{L}^2}.$$

Lemma 2.7 is proved. □

## 3. Estimate for the derivative

First we state the local existence of solutions to the Cauchy problem (1.1) in the functional space $\mathbf{H}^1 \cap \mathbf{H}^{0,1}$ (see [2] for the proof.)

**Theorem 3.1.** *Assume that the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$. Then there exists a time $T > 0$ which depends on the norm $\|u_0\|_{\mathbf{H}^1\cap\mathbf{H}^{0,1}}$ such that the Cauchy problem* (1.2) *has a unique solution $\mathcal{U}(-t)u \in \mathbf{C}\left([0,T];\mathbf{H}^1\cap\mathbf{H}^{0,1}\right)$. Existence time $T$ can be taken large if the norm $\|u_0\|_{\mathbf{H}^1\cap\mathbf{H}^{0,1}}$ is small.*

Define the norm

$$\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_T} = \sup_{t\in[0,T]}\left(\left\|\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty} + Q^{\frac{1}{2}}(t)\left\|\left\langle\xi\right\rangle^{-\gamma}\widehat{\varphi}(t)\right\|_{\mathbf{L}^\infty}\right),$$

where $Q(t) = 1 + \delta^2\log(1+t)$, $\delta = \varepsilon^{1+\gamma}$, $\gamma > 0$ is small.

**Lemma 3.2.** *Let $\widehat{u_0} \in \mathbf{H}^1$ and $\left\|\widehat{u_0}\right\|_{\mathbf{H}^1} \leq \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Also we suppose that $\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_T} \leq 4\varepsilon$. Then the solutions $\widehat{\varphi}$ of* (2.2) *satisfy the estimate $\left\|\widehat{\varphi}_\xi(t)\right\|_{\mathbf{L}^2} < C\varepsilon K(t)$ for all $t\in[0,T]$, where $K(t) = 1 + \varepsilon^2 t^{\frac{1}{4}}Q^{-\frac{3}{2}}(t)$.*

Proof. Arguing by contradiction in view of continuity we can find a time $\widetilde{T} \in (0,T]$ such that $\left\|\widehat{\varphi}_\xi(t)\right\|_{\mathbf{L}^2} \leq C\varepsilon K(t)$ for all $t \in \left[0,\widetilde{T}\right]$. Differentiating equation (2.2) with respect to $\xi$, we get

$$i\partial_t\widehat{\varphi}_\xi = i\Omega' e^{it\Omega}\mathcal{D}_3\mathcal{Q}^*(3t)\psi^3 + \frac{1}{3t}e^{it\Omega}\mathcal{D}_3\partial_\xi\mathcal{Q}^*(3t)\psi^3.$$

We transform the first summand on the right-hand side of the above equation using formula (2.3) with $k = 0$. Hence for the new dependent variable $g = \widehat{\varphi}_\xi - I_0$, we get $i\partial_t g = \sum_{j=1}^7 I_j$, where $I_0 = t\Omega'\mathcal{Q}^*M^2H_1\psi^3$,

$$I_1 = -i\Omega'\mathcal{Q}^*M^2H_3\psi^3,\ I_2 = i\Omega'\mathcal{Q}^*M^2H_5\psi^2\psi_2,\ I_3 = -3\Omega'\mathcal{Q}^*M^4H_1\psi^5,$$
$$I_4 = \frac{1}{3t}e^{it\Omega}\mathcal{D}_3\partial_\xi\mathcal{Q}^*(3t)\psi^3,\ I_5 = it\Omega'\mathcal{Q}^*M^2H_4\psi^2\left(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2\right),$$
$$I_6 = -\frac{3}{2}t\Omega'\mathcal{Q}^*M^2H_1\psi\psi_1\left(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2\right),$$
$$I_7 = -\frac{3}{4}it\Omega'\mathcal{Q}^*M^2\eta H_1\psi^2\left(\left(2+3\eta^2\right)\mathcal{A}_0\psi - \eta\mathcal{A}_0\psi_1 - \mathcal{A}_0\psi_2\right).$$

By Lemma 2.3 we find $\left\|\langle\eta\rangle^{3-j}t\mathcal{A}_0\psi_j\right\|_{\mathbf{L}^2} \leq C\left\|\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} + C\left|\widehat{\varphi}(0)\right| \leq C\varepsilon K$, for $j = 0,1,2$. Next by Lemma 2.2 and estimate $\left|A_j(t,\eta)\right| \leq \langle\eta\rangle^{j-1}$ we find

$$\left\|\langle\eta\rangle^{1-j}\psi_j\right\|_{\mathbf{L}^\infty} \leq \left\|\langle\eta\rangle^{1-j}A_j\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + \left\|\langle\eta\rangle^{1-j}\left(\psi_j - A_j\widehat{\varphi}\right)\right\|_{\mathbf{L}^\infty}$$
$$\leq \left\|\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + Ct^{-\frac{1}{4}}\left\|\widehat{\varphi}_\xi\right\|_{\mathbf{L}^2} \leq C\varepsilon\left(1 + t^{-\frac{1}{4}}K(t)\right) \leq C\varepsilon$$

for $j = 0, 1, 2$, and, similarly,

$$\left\|\langle\eta\rangle^{1-j}\langle\widetilde{\eta}\rangle^{-\gamma}\psi_j\right\|_{\mathbf{L}^\infty} = \left\|\langle\eta\rangle^{1-j}\langle\widetilde{\eta}\rangle^{-\gamma}A_j\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + \left\|\langle\eta\rangle^{1-j}\langle\widetilde{\eta}\rangle^{-\gamma}\left(\psi_j - A_j\widehat{\varphi}\right)\right\|_{\mathbf{L}^\infty}$$
$$\leq C\left\|\langle\widetilde{\eta}\rangle^{-\gamma}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + Ct^{-\frac{1}{4}}\left\|\widehat{\varphi}_\xi\right\|_{\mathbf{L}^2} \leq C\varepsilon\left(Q^{-\frac{1}{2}}(t) + t^{-\frac{1}{4}}K(t)\right) \leq C\varepsilon Q^{-\frac{1}{2}}(t)$$

for all $t \in \left[0, \widetilde{T}\right]$. In particular, we obtain

$$\left\|\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-1}\psi^3\right\|_{\mathbf{L}^2} \leq C\left\|\langle\widetilde{\eta}\rangle^{3\gamma-1}\right\|_{\mathbf{L}^2}\left\|\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-3\gamma}\psi\right\|^3_{\mathbf{L}^\infty} \leq C\varepsilon^3 t^{-\frac{1}{4}}Q^{-\frac{3}{2}}(t).$$

Note that $tS_3(\xi,\eta) \geq \frac{1}{8}\left(\widetilde{\xi}^2\langle\xi\rangle^2 + \widetilde{\eta}^2\langle\eta\rangle^2\right)$, so the kernel $\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^2 H_1(t,\xi,\eta)\langle\widetilde{\eta}\rangle = \frac{\langle\widetilde{\xi}\rangle\langle\xi\rangle^2\langle\widetilde{\eta}\rangle}{1+itS_3(\xi,\eta)}$ satisfies the estimates $\left|\partial_\eta^m\partial_\xi^n\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^2 H_1(t,\xi,\eta)\langle\widetilde{\eta}\rangle\right| \leq Ct^{\frac{m+n}{2}}$ for all $\xi,\eta \in \mathbb{R}$, $t \geq 1$, with $m, n = 0, 1$. Hence application of Lemma 2.7 yields

$$\|I_0\|_{\mathbf{L}^2} = \left\|t\Omega' Q^* M^2 H_1\psi^3\right\|_{\mathbf{L}^2} \leq Ct^{\frac{1}{2}}\left\|\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^2 Q^* M^2 H_1\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-1}\psi^3\right\|_{\mathbf{L}^2}$$
$$\leq Ct^{\frac{1}{2}}\left\|\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-1}\psi^3\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{\frac{1}{4}}Q^{-\frac{3}{2}}(t) \leq C\varepsilon^3 K.$$

Since $H_3 = t\partial_t H_1 + 2H_1 - \frac{3}{2\Lambda''(\eta)}H_1$, then as above by Lemma 2.7, we get

$$\|I_1\|_{\mathbf{L}^2} = \left\|\Omega' Q^* M^2 H_3\psi^3\right\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{2}}\left\|\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^2 Q^* M^2 H_3\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-1}\psi^3\right\|_{\mathbf{L}^2}$$
$$\leq Ct^{-\frac{1}{2}}\left\|\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-1}\psi^3\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1}K,$$
$$\|I_2\|_{\mathbf{L}^2} = \frac{3}{4}\left\|\Omega' Q^* M^2 H_1\frac{1}{\Lambda''(\eta)}\psi^2\psi_2\right\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{2}}\left\|\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^2 Q^* M^2 H_1\langle\widetilde{\eta}\rangle\frac{1}{\Lambda''(\eta)}\langle\widetilde{\eta}\rangle^{-1}\psi^2\psi_2\right\|_{\mathbf{L}^2}$$
$$\leq Ct^{-\frac{1}{2}}\left\|\langle\eta\rangle^{-1}\langle\widetilde{\eta}\rangle^{-1}\psi^2\psi_2\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1}K$$

and

$$\|I_3\|_{\mathbf{L}^2} = 3\left\|\Omega' Q^* M^4 H_1\psi^5\right\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{2}}\left\|\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^2 Q^* M^4 H_1\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-1}\psi^5\right\|_{\mathbf{L}^2}$$
$$\leq Ct^{-\frac{1}{2}}\left\|\langle\eta\rangle\langle\widetilde{\eta}\rangle^{-1}\psi^5\right\|_{\mathbf{L}^2} \leq C\varepsilon^5 t^{-1}K.$$

Next we consider the term $I_4$. Since $\partial_\xi S(\xi,\eta) = (\xi-\eta)\left(1 + \xi^2 + \eta\xi + \eta^2\right)$, we find

$$\partial_\xi Q^*\phi = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)\,it\partial_\xi S(\xi,\eta)\,d\eta$$
$$= ti\xi Q^*\phi - tQ^* i\eta\phi - t(i\xi)^3 Q^*\phi + tQ^*(i\eta)^3\phi$$
$$= tQ^*\mathcal{A}_0\phi - tQ^*\left(\mathcal{A}_0\mathcal{A}_1^2 + i\eta\mathcal{A}_0\mathcal{A}_1 - \eta^2\mathcal{A}_0\right)\phi.$$

Using the identity $\mathcal{A}_1(3t)\psi^3 = 3\psi^2\mathcal{A}_1(t)\psi = 3\psi^2\psi_1$, we obtain

$$\|I_4\|_{\mathbf{L}^2} = \frac{1}{3t}\left\|\partial_\xi Q^*(3t)\psi^3\right\|_{\mathbf{L}^2} \leq C\left\|\langle\eta\rangle\mathcal{A}_0(3t)\psi^3\right\|_{\mathbf{L}^2} + C\left\|\langle\eta\rangle\mathcal{A}_0(3t)\psi^2\psi_2\right\|_{\mathbf{L}^2}$$
$$+ C\left\|\langle\eta\rangle\mathcal{A}_0(3t)\psi_1^2\psi\right\|_{\mathbf{L}^2} + C\left\|\langle\eta\rangle\eta\mathcal{A}_0(3t)\psi^2\psi_1\right\|_{\mathbf{L}^2} + C\left\|\langle\eta\rangle\eta^2\mathcal{A}_0(3t)\psi^3\right\|_{\mathbf{L}^2}$$
$$\leq C\left(\sum_{j=0}^{2}\left\|\langle\eta\rangle^{1-j}\psi_j\right\|_{\mathbf{L}^\infty}\right)^2\sum_{j=0}^{2}\left\|\langle\eta\rangle^{3-j}\mathcal{A}_0\psi_j\right\|_{\mathbf{L}^2} \leq C\varepsilon^2 t^{-1}\left\|\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1}K.$$

Since the kernel $t^{\frac{1}{2}}\left\langle\widetilde{\xi}\right\rangle\langle\xi\rangle^2 H_4(t,\xi,\eta)$, with $H_4 = \frac{3}{4}(\xi H_1 + i\mathcal{A}_0 H_1)$ satisfies the estimates

$\left|\partial_\eta^m \partial_\xi^n t^{\frac{1}{2}} \left\langle \widetilde{\xi} \right\rangle \langle \xi \rangle^2 H_4(t,\xi,\eta)\right| \leq C t^{\frac{m+n}{2}}$ for all $\xi, \eta \in \mathbb{R}$, $t \geq 1$, where $m, n = 0, 1$, by Lemma 2.7 we get

$$\begin{aligned}
&\|I_5\|_{\mathbf{L}^2} = t \left\|\Omega' Q^* M^2 H_4 \psi^2 (2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2)\right\|_{\mathbf{L}^2} \\
&\leq C t^{\frac{1}{2}} \left\| \left\langle \widetilde{\xi} \right\rangle \langle \xi \rangle^2 Q^* M^2 H_4 \psi^2 (2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2) \right\|_{\mathbf{L}^2} \\
&\leq C \left\|\langle \eta \rangle \psi^2 \mathcal{A}_0 \psi\right\|_{\mathbf{L}^2} + C \left\|\langle \eta \rangle \psi^2 \mathcal{A}_0 \psi_2\right\|_{\mathbf{L}^2} \leq C \|\langle \eta \rangle \psi\|_{\mathbf{L}^\infty}^2 \sum_{j=0}^{2} \left\|\langle \eta \rangle^{3-j} \mathcal{A}_0 \psi_j\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1} K.
\end{aligned}$$

Next we represent $\psi_1 = \mathcal{Q}\xi\widehat{\varphi} = -i\mathcal{A}_1 \mathcal{Q}\widehat{\varphi} = \eta\psi - i\mathcal{A}_0\psi$. Hence $I_6 = I_8 + I_9$, where

$$\begin{aligned}
I_8 &= -\frac{3}{2} t \Omega' Q^* M^2 H_1 \eta \psi^2 (2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2), \\
I_9 &= \frac{3i}{2} t \Omega' Q^* M^2 H_1 \psi (\mathcal{A}_0\psi)(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2).
\end{aligned}$$

The term $I_8$ is estimated as $I_5$

$$\begin{aligned}
\|I_8\|_{\mathbf{L}^2} &= \frac{3}{2} t \left\|\Omega' Q^* M^2 H_1 \eta \psi^2 (2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2)\right\|_{\mathbf{L}^2} \\
&\leq C t^{\frac{1}{2}} \left\| \left\langle \widetilde{\xi} \right\rangle \langle \xi \rangle^2 Q^* M^2 H_1 \eta \psi^2 (2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2) \right\|_{\mathbf{L}^2} \\
&\leq C \left\|\langle \eta \rangle \psi^2 \mathcal{A}_0 \psi\right\|_{\mathbf{L}^2} + C \left\|\langle \eta \rangle \psi^2 \mathcal{A}_0 \psi_2\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1} K.
\end{aligned}$$

To estimate $I_9$ we apply Lemma 2.4 with $j = 1$

$$\begin{aligned}
&\|I_9\|_{\mathbf{L}^2} = \frac{3}{2} t \left\|\Omega' Q^* M^2 H_1 \psi (\mathcal{A}_0\psi)(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2)\right\|_{\mathbf{L}^2} \\
&\leq C t^{\frac{1}{2}} \left\| \left\langle \widetilde{\xi} \right\rangle^{-1} \right\|_{\mathbf{L}^2} \left\| \left\langle \widetilde{\xi} \right\rangle^2 \langle \xi \rangle^2 Q^* M^2 H_1 \psi (\mathcal{A}_0\psi)(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2) \right\|_{\mathbf{L}^\infty} \\
&\leq C t^{\frac{3}{4}} \left\|\langle \eta \rangle^2 \psi (\mathcal{A}_0\psi)(2\mathcal{A}_0\psi + \mathcal{A}_0\psi_2)\right\|_{\mathbf{L}^1} \\
&\leq C t^{\frac{3}{4}} \|\langle \eta \rangle \psi\|_{\mathbf{L}^\infty} \left\|\langle \eta \rangle^3 \mathcal{A}_0\psi\right\|_{\mathbf{L}^2} \sum_{j=0}^{2} \left\|\langle \eta \rangle^{3-j} \mathcal{A}_0 \psi_j\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-\frac{5}{4}} K^2 \leq C\varepsilon^3 t^{-1} K.
\end{aligned}$$

Finally, we find

$$\begin{aligned}
&\|I_7\|_{\mathbf{L}^2} = \frac{3}{4} t \left\|\Omega' Q^* M^2 \eta H_1 \psi^2 \left(\left(2 + 3\eta^2\right) \mathcal{A}_0\psi - \eta\mathcal{A}_0\psi_1 - \mathcal{A}_0\psi_2\right)\right\|_{\mathbf{L}^2} \\
&\leq C t^{\frac{1}{2}} \left\| \left\langle \widetilde{\xi} \right\rangle Q^* M^2 H_1 \eta \psi^2 \left(\left(2 + 3\eta^2\right) \mathcal{A}_0\psi - \eta\mathcal{A}_0\psi_1 - \mathcal{A}_0\psi_2\right) \right\|_{\mathbf{L}^2} \\
&\leq C \left\|\langle \eta \rangle^3 \psi^2 \mathcal{A}_0 \psi\right\|_{\mathbf{L}^2} + C \left\|\langle \eta \rangle^2 \psi^2 \mathcal{A}_0 \psi_1\right\|_{\mathbf{L}^2} + C \left\|\langle \eta \rangle \psi^2 \mathcal{A}_0 \psi_2\right\|_{\mathbf{L}^2} \\
&\leq C \|\langle \eta \rangle \psi\|_{\mathbf{L}^\infty}^2 \sum_{j=0}^{2} \left\|\langle \eta \rangle^{3-j} \mathcal{A}_0 \psi_j\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1} K.
\end{aligned}$$

Therefore we obtain the inequality $\frac{d}{dt} \|g\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1} K$ for all $t \in \left[0, \widetilde{T}\right]$. Integrating this inequality we find the estimate

$$\|g(t)\|_{\mathbf{L}^2} \leq \|g(1)\|_{\mathbf{L}^2} + C\varepsilon^3 \int_1^t \tau^{-1} d\tau + C\varepsilon^5 \int_1^t \tau^{-\frac{3}{4}} Q^{-\frac{3}{2}}(\tau)\, d\tau$$

$$\leq C\varepsilon + C\varepsilon^3 \log(t+1) + C\varepsilon^5 \left( \int_1^{\sqrt{t}} + \int_{\sqrt{t}}^t \right) \frac{\tau^{-\frac{3}{4}} d\tau}{\left(1 + \delta^2 \log(1+\tau)\right)^{\frac{3}{2}}}$$

$$\leq C\varepsilon + C\varepsilon^3 \log(t+1) + C\varepsilon^5 \int_1^{\sqrt{t}} \tau^{-\frac{3}{4}} d\tau + C\varepsilon^5 Q^{-\frac{3}{2}} \left(\sqrt{t}\right) \int_{\sqrt{t}}^t \tau^{-\frac{3}{4}} d\tau$$

$$\leq C\varepsilon + C\varepsilon^3 t^{\frac{1}{8}} + C\varepsilon^5 t^{\frac{1}{4}} Q^{-\frac{3}{2}} \left(\sqrt{t}\right) \leq C\varepsilon + C\varepsilon^3 t^{\frac{1}{4}} Q^{-\frac{3}{2}}(t) < C\varepsilon K$$

for all $t \in \left[0, \widetilde{T}\right]$, since $t^{\frac{1}{8}} \leq C t^{\frac{1}{4}} Q^{-\frac{3}{2}}(t)$, i.e. $Q(t) = 1 + \delta^2 \log(1+t) \leq C t^{\frac{1}{12}}$. Therefore it follows that $\left\|\widehat{\varphi}_\xi\right\|_{\mathbf{L}^2} < CK$ for all $t \in \left[0, \widetilde{T}\right]$. We arrive to a contradiction. Lemma 3.2 is proved. □

## 4. Estimate for the uniform norm

First we obtain a new equation governing large time asymptotics of solutions $\widehat{\varphi}(t)$ of equation (2.2). Define the norm

$$\left\|\widehat{\varphi}\right\|_{\mathbf{X}_T} = \left\|\widehat{\varphi}\right\|_{\mathbf{Z}_T} + \sup_{t \in [0,T]} \frac{1}{K(t)} \left\|\widehat{\varphi}_\xi(t)\right\|_{\mathbf{L}^2},$$

where $K(t) = 1 + \varepsilon^2 t^{\frac{1}{4}} Q^{-\frac{3}{2}}(t)$, $Q(t) = 1 + \delta^2 \log(1+t)$, $\delta = \varepsilon^{1+\gamma}$, $\gamma > 0$ is small.

**Lemma 4.1.** *Let the function* $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t) u(t)$ *satisfy equation* (2.2). *Suppose that* $\left\|\widehat{\varphi}\right\|_{\mathbf{X}_T} \leq C\varepsilon$. *Then for* $y = \widehat{\varphi} + iQ^* M^2 H_1 \psi^3$ *the following equation is true* $\partial_t y = -3^{-\frac{1}{2}} \Phi t^{-1} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\alpha}{2}} y^3 + h$ *for* $t \geq 1$, *where*

$$h = O\left(\varepsilon^3 t^{-1} \left(t^{-\frac{1}{4}} + \varepsilon^2 Q^{-2}(t) + \left\{\widetilde{\xi}\right\}^\gamma \left\langle \widetilde{\xi} \right\rangle^{-\gamma} Q^{-\frac{3}{2}}(t)\right)\right).$$

Proof. By equation (2.2) and formula (2.3) with $k = -1$, we obtain for $y = \widehat{\varphi} + I_0$ a new equation $i y_t = \sum_{j=1}^6 I_j$, where $I_0 = iQ^* M^2 H_1 \psi^3$, $I_1 = -t^{-1} Q^* M^2 H_3 \psi^3$, $I_2 = t^{-1} Q^* M^2 H_5 \psi^2 \psi_2$, $I_3 = 3it^{-1} Q^* M^4 H_1 \psi^5$,

$$I_4 = Q^* M^2 H_4 \psi^2 \left(2\mathcal{A}_0 \psi + \mathcal{A}_0 \psi_2\right),$$

$$I_5 = -\frac{3}{4} Q^* M^2 \eta H_1 \psi^2 \left(\left(2 + 3\eta^2\right) \mathcal{A}_0 \psi - \eta \mathcal{A}_0 \psi_1 - \mathcal{A}_0 \psi_2\right),$$

and

$$I_6 = \frac{3i}{2} Q^* M^2 H_1 \psi \psi_1 \left(2\mathcal{A}_0 \psi + \mathcal{A}_0 \psi_2\right).$$

By formula (2.1) with $k = 2, 4$ we get $I_1 = -t^{-1} e^{it\Omega} \mathcal{D}_3 Q^*(3t) H_3 \psi^3$, $I_2 = t^{-1} e^{it\Omega} \mathcal{D}_3 Q^*(3t) H_5 \psi^2 \psi_2$, $I_3 = 3it^{-1} e^{it\Omega_5} \mathcal{D}_4 Q^*(5t) H_1 \psi^5$. Then in view of Lemma 2.5 we find

$$I_1 = -t^{-1} e^{it\Omega} \mathcal{D}_3 A^*_{H_3}(3t) \psi^3 + O\left(t^{-\frac{5}{4}} \left\|\langle\widetilde{\eta}\rangle^{-\alpha} \partial_\eta \psi^3\right\|_{\mathbf{L}^2}\right),$$

$$I_2 = t^{-1} e^{it\Omega} \mathcal{D}_3 A^*_{H_5}(3t) \psi^2 \psi_2 + O\left(t^{-\frac{5}{4}} \left\|\langle\widetilde{\eta}\rangle^{-\alpha} \partial_\eta \psi^2 \psi_2\right\|_{\mathbf{L}^2}\right)$$

and

$$I_3 = 3it^{-1} e^{it\Omega_5} \mathcal{D}_5 A^*_{H_1}(5t) \psi^5 + O\left(t^{-\frac{5}{4}} \left\|\langle\widetilde{\eta}\rangle^{-\alpha} \partial_\eta \psi^5\right\|_{\mathbf{L}^2}\right),$$

where $\alpha \in \left(0, \frac{1}{2}\right)$. Application of Lemma 2.3 yields $\left\| \langle \eta \rangle^{1-j} \partial_\eta \psi_j \right\|_{\mathbf{L}^2} \le C \left\| \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} + C \left| \widehat{\varphi}(0) \right| \le C\varepsilon K$ for $j = 0, 1, 2$. Next by Lemma 2.2 and estimate $\left| A_j(t, \eta) \right| \le \langle \eta \rangle^{j-1}$ we find

$$\begin{aligned} &\left\| \langle \eta \rangle^{1-j} \langle \widetilde{\eta} \rangle^{-\gamma} \psi_j \right\|_{\mathbf{L}^\infty} = \left\| \langle \eta \rangle^{1-j} \langle \widetilde{\eta} \rangle^{-\gamma} A_j \widehat{\varphi} \right\|_{\mathbf{L}^\infty} + \left\| \langle \eta \rangle^{1-j} \langle \widetilde{\eta} \rangle^{-\gamma} \left( \psi_j - A_j \widehat{\varphi} \right) \right\|_{\mathbf{L}^\infty} \\ &\le C \left\| \langle \widetilde{\eta} \rangle^{-\gamma} \widehat{\varphi} \right\|_{\mathbf{L}^\infty} + C t^{-\frac{1}{4}} \left\| \widehat{\varphi}_\xi \right\|_{\mathbf{L}^2} \le C\varepsilon \left( Q^{-\frac{1}{2}}(t) + t^{-\frac{1}{4}} K(t) \right) \le C\varepsilon Q^{-\frac{1}{2}}(t) \end{aligned}$$

for all $t \in [0, T]$. Therefore

$$\left\| \langle \widetilde{\eta} \rangle^{-2\gamma} \partial_\eta \psi^3 \right\|_{\mathbf{L}^2} \le \left\| \langle \eta \rangle \langle \widetilde{\eta} \rangle^{-\gamma} \psi \right\|_{\mathbf{L}^\infty}^2 \left\| \langle \eta \rangle \partial_\eta \psi \right\|_{\mathbf{L}^2} \le C\varepsilon^3 Q^{-1} K,$$

$$\begin{aligned} &\left\| \langle \widetilde{\eta} \rangle^{-2\gamma} \partial_\eta \psi^2 \psi_2 \right\|_{\mathbf{L}^2} \le \left\| \langle \eta \rangle \langle \widetilde{\eta} \rangle^{-\gamma} \psi \right\|_{\mathbf{L}^\infty}^2 \left\| \langle \eta \rangle^{-1} \partial_\eta \psi_2 \right\|_{\mathbf{L}^2} \\ &+ \left\| \langle \eta \rangle \langle \widetilde{\eta} \rangle^{-\gamma} \psi \right\|_{\mathbf{L}^\infty} \left\| \langle \eta \rangle^{-1} \langle \widetilde{\eta} \rangle^{-\gamma} \psi_2 \right\|_{\mathbf{L}^\infty} \left\| \langle \eta \rangle \partial_\eta \psi \right\|_{\mathbf{L}^2} \le C\varepsilon^3 Q^{-1} K \end{aligned}$$

and

$$\left\| \langle \widetilde{\eta} \rangle^{-4\gamma} \partial_\eta \psi^5 \right\|_{\mathbf{L}^2} \le \left\| \langle \eta \rangle \langle \widetilde{\eta} \rangle^{-\gamma} \psi \right\|_{\mathbf{L}^\infty}^4 \left\| \langle \eta \rangle \partial_\eta \psi \right\|_{\mathbf{L}^2} \le C\varepsilon^5 Q^{-2} K.$$

By Lemma 2.2 we get $\psi_j = A_j \widehat{\varphi} + O\left( \varepsilon \langle \eta \rangle^{j-\frac{3}{2}} t^{-\frac{1}{4}} K \right)$. Thus by Lemma 2.6 we find

$$I_1 = \sqrt{i} \Phi t^{-1} e^{it\Omega} \mathcal{D}_3 \left\langle \widetilde{\xi} \right\rangle^{-\frac{\alpha}{2}} A_0^3 \widehat{\varphi}^3 + O\left( \varepsilon^3 t^{-\frac{5}{4}} Q^{-1} K \right) = -\frac{i\Phi}{t\sqrt{3} \left\langle \widetilde{\xi} \right\rangle^{\frac{\alpha}{2}}} \widehat{\varphi}^3 + h,$$

$$I_2 = t^{-1} e^{it\Omega} \mathcal{D}_3 A_{H_5}^* (3t, \xi) \psi^2 \psi_2 + O\left( \varepsilon^3 t^{-\frac{5}{4}} Q^{-1} K \right) \le h$$

and

$$I_3 = 3it^{-1} e^{it\Omega_5} \mathcal{D}_5 A_{H_1}^* (5t, \xi) \psi^5 + O\left( \varepsilon^3 t^{-\frac{5}{4}} Q^{-1} K \right) \le h,$$

since $\left\| e^{it\Omega} \mathcal{D}_3 \left\langle \widetilde{\xi} \right\rangle^{-\frac{\alpha}{2}} \widehat{\varphi}^3 - 3^{-\frac{1}{2}} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\alpha}{2}} \widehat{\varphi}^3 (t, \xi) \right\|_{\mathbf{L}^\infty} \le h$. Next we estimate by Lemma 2.4

$$\begin{aligned} &\|I_4\|_{\mathbf{L}^\infty} + \|I_5\|_{\mathbf{L}^\infty} \le C \left\| \langle \eta \rangle^2 \langle \widetilde{\eta} \rangle^{-1} \psi^2 \left( 2\mathcal{A}_0 \psi + \mathcal{A}_0 \psi_2 \right) \right\|_{\mathbf{L}^1} \\ &+ C \left\| \langle \eta \rangle^2 \langle \widetilde{\eta} \rangle^{-1} \psi^2 \left( \left( 2 + 3\eta^2 \right) \mathcal{A}_0 \psi - \eta \mathcal{A}_0 \psi_1 - \mathcal{A}_0 \psi_2 \right) \right\|_{\mathbf{L}^1} \\ &\le C t^{-1} \left\| \langle \eta \rangle \langle \widetilde{\eta} \rangle^{-\gamma} \psi \right\|_{\mathbf{L}^\infty}^2 \left\| \langle \widetilde{\eta} \rangle^{2\gamma - 1} \right\|_{\mathbf{L}^2} \sum_{j=0}^{2} \left\| \langle \eta \rangle^{1-j} \partial_\eta \psi_j \right\|_{\mathbf{L}^2} \le C\varepsilon^3 t^{-\frac{5}{4}} Q^{-1} K \le h. \end{aligned}$$

Finally we represent $\psi_1 = \mathcal{Q} \xi \widehat{\varphi} = -i\mathcal{A}_1 \mathcal{Q} \widehat{\varphi} = \eta \psi - i\mathcal{A}_0 \psi$. Hence $I_6 = I_7 + I_8$, where

$$\begin{aligned} I_7 &= \frac{3i}{2} \mathcal{Q}^* M^2 H_1 \eta \psi^2 \left( 2\mathcal{A}_0 \psi + \mathcal{A}_0 \psi_2 \right), \\ I_8 &= \frac{3}{2} \mathcal{Q}^* M^2 H_1 \psi \left( \mathcal{A}_0 \psi \right) \left( 2\mathcal{A}_0 \psi + \mathcal{A}_0 \psi_2 \right). \end{aligned}$$

As above by Lemma 2.4 we find $\|I_7\|_{\mathbf{L}^\infty} \le C \left\| \langle \eta \rangle^2 \langle \widetilde{\eta} \rangle^{-1} \psi^2 \left( 2\mathcal{A}_0 \psi + \mathcal{A}_0 \psi_2 \right) \right\|_{\mathbf{L}^1} \le h$. Also we have

$$\begin{aligned} &\|I_8\|_{\mathbf{L}^\infty} \le C t^{\frac{1}{2}} \left\| \langle \eta \rangle^2 \langle \widetilde{\eta} \rangle^{-2} \psi \left( \mathcal{A}_0 \psi \right) \left( 2\mathcal{A}_0 \psi + \mathcal{A}_0 \psi_2 \right) \right\|_{\mathbf{L}^1} \\ &\le C t^{-\frac{3}{2}} \|\langle \eta \rangle \psi\|_{\mathbf{L}^\infty} \left\| \partial_\eta \psi \right\|_{\mathbf{L}^2} \left( \left\| \partial_\eta \psi \right\|_{\mathbf{L}^2} + \left\| \langle \eta \rangle^{-1} \partial_\eta \psi_2 \right\|_{\mathbf{L}^2} \right) \le C\varepsilon^3 t^{-\frac{3}{2}} K^2 \le h. \end{aligned}$$

Thus we get the equation $\partial_t y = -3^{-\frac{1}{2}}\Phi t^{-1}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}}\widehat{\varphi}^3 + h$. By Lemma 2.4

$$\|I_0\|_{\mathbf{L}^\infty} \le Ct^{\frac{1}{2}}\left\|\langle\eta\rangle^2\left\langle\widetilde{\eta}\right\rangle^{-2}\psi^3\right\|_{\mathbf{L}^1} \le Ct^{\frac{1}{2}}\left\|\left\langle\widetilde{\eta}\right\rangle^{3\gamma-2}\right\|_{\mathbf{L}^1}\left\|\langle\eta\rangle\left\langle\widetilde{\eta}\right\rangle^{-\gamma}\psi\right\|^3_{\mathbf{L}^\infty} \le C\varepsilon^3 Q^{-\frac{3}{2}}.$$

Hence $\widehat{\varphi} = y + O\left(\varepsilon^3 Q^{-\frac{3}{2}}\right)$. Therefore we obtain the result of the lemma. Lemma 4.1 is proved. □

We consider now the Cauchy problem for the ordinary differential equation depending on the parameter $\xi \in \mathbb{R}$

$$\begin{cases} y_t = -C_0 t^{-1}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}} y^3 + h,\ t \ge 1, \\ y(1,\xi) = y_1(\xi), \end{cases} \tag{4.1}$$

where $C_0 = 3^{-\frac{1}{2}}\Phi > 0$ and $h = O\left(\varepsilon^3 t^{-1}\left(t^{-\frac{1}{4}} + \varepsilon^2 Q^{-2}(t) + \left\{\widetilde{\xi}\right\}^{\gamma}\left\langle\widetilde{\xi}\right\rangle^{-\gamma} Q^{-\frac{3}{2}}(t)\right)\right)$, $Q(t) = 1 + \delta^2\log(1+t)$.

**Lemma 4.2.** *Suppose that there exists a solution $y \in \mathbf{C}([1,T]\times\mathbb{R})$ of* (4.1). *Let the initial data $y_1$ satisfy the following conditions $\delta \le |y_1(\xi)| \le \varepsilon$, $\left|\arg y_1(\xi)\right| \le \frac{\pi}{8}$ for $|\xi| \le 1$, where $\varepsilon > 0$ is sufficiently small, and $\delta = \varepsilon^{1+\gamma}$. Then the estimates are true*

$$|y(t,\xi)| \le 2\varepsilon\left(1 + C_0\delta^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{-\frac{1}{2}}$$

*and*

$$\left|\arg y(t,\xi)\right| \le \frac{\pi}{4}\left(1 + C_0\delta^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{-\frac{1}{8}}$$

*for all $t > 1$ and $\xi \in \mathbb{R}$.*

Proof. For the case of $|\xi| > 1$ we have $\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}} \le (1+t)^{-\frac{\alpha}{4}}$, then by (4.1) we get $y_t = O\left(\varepsilon^3 t^{-1-\gamma}\right) + O\left(\varepsilon^5 t^{-1} Q^3(t)\right)$. Integration with respect to time yields $|y(t)| \le \varepsilon + C\varepsilon^3$. So next we consider the case of $|\xi| \le 1$. We change the dependent variable $y = re^{i\omega}$, where $r > 0$ and $\omega$ is a real-valued function. Then from equation (4.1) we get $r_t + ir\omega_t = -C_0 t^{-1}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}} r^3 e^{2i\omega} + he^{-i\omega}$. Hence taking the real and imaginary part we obtain

$$r_t = -C_0 t^{-1}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}} r^3\cos 2\omega + \operatorname{Re} he^{-i\omega} \tag{4.2}$$

and

$$\omega_t = -C_0 t^{-1}\left\langle\widetilde{\xi}\right\rangle^{-\frac{\alpha}{2}} r^2\sin 2\omega + \operatorname{Im} hr^{-1}e^{-i\omega}, \tag{4.3}$$

with the initial conditions $r(1,\xi) = |y_1(\xi)|$, $\omega(1,\xi) = \arg y_1(\xi)$. Let us prove the following estimates

$$\frac{1}{2}\left(1 + C_0\delta^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right) < \frac{|y_1|^2}{r^2(t)} < 2\left(1 + 2C_0\varepsilon^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)$$

and $|\omega(t,\xi)| < \frac{\pi}{4}$ for all $t \in [1,T]$, $|\xi| \le 1$. By a contradiction we suppose that there exists a maximal time $\widetilde{T} \in (1,T]$, such that

$$\frac{1}{2}\left(1 + C_0\delta^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right) \le \frac{|y_1|^2}{r^2(t)} \le 2\left(1 + 2C_0\varepsilon^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right) \tag{4.4}$$

and $|\omega(t,\xi)| \le \frac{\pi}{4}$ for all $t \in \left[1, \widetilde{T}\right], |\xi| \le 1$. Dividing equation (4.2) by $r^3$, we find

$$C_0 t^{-1} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\alpha}{2}} - 2r^{-3}|h| \le \partial_t r^{-2} \le 2C_0 t^{-1} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\alpha}{2}} + 2r^{-3}|h|.$$

Then integration in time yields $\frac{|y_1|^2}{r^2(t)} \le 1 + 2C_0|y_1|^2 \int_1^t \tau^{-1} \left\langle \sqrt{\tau}\xi \right\rangle^{-\frac{\alpha}{2}} d\tau + 2\varepsilon^2 \int_1^t r^{-3}|h|\, d\tau$ and $\frac{|y_1|^2}{r^2(t)} \ge 1 + C_0|y_1|^2 \int_1^t \tau^{-1} \left\langle \sqrt{\tau}\xi \right\rangle^{-\frac{\alpha}{2}} d\tau - 2\varepsilon^2 \int_1^t r^{-3}|h|\, d\tau$. We have $\int_1^t \tau^{-1} \left\langle \sqrt{\tau}\xi \right\rangle^{-\frac{\alpha}{2}} d\tau = \log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right) + O(1)$. By the assumptions on $h$ and (4.4) we obtain $Q(t) = 1 + \delta^2 \log(1+t)$

$$\begin{aligned}\varepsilon^2 \int_1^t r^{-3}|h|\, d\tau &\le C\varepsilon^{2-3\gamma} \int_1^t \tau^{\gamma-\frac{5}{4}} d\tau + C\varepsilon^{2-6\gamma} \int_1^t \frac{\left|\xi\sqrt{\tau}\right|^{\gamma} d\tau}{\tau\left\langle \xi\sqrt{\tau}\right\rangle^{2\gamma}} \\ &+ C\varepsilon^{4-5\gamma}\left(1 + \varepsilon^2 \log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right)\right)^{\frac{3}{4}} \int_1^t \frac{d\tau}{\tau\left(1+\delta^2\log(1+\tau)\right)^{\frac{5}{4}}} \\ &\le C\varepsilon^{2-7\gamma} + C\varepsilon^{\frac{7}{2}-7\gamma}\left(\log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right)\right)^{\frac{3}{4}}\end{aligned}$$

for all $t \in \left[1, \widetilde{T}\right], |\xi| \le 1$. Therefore we get

$$\frac{1}{2}\left(1 + C_0\delta^2 \log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right)\right) < \frac{|y_1|^2}{r^2(t)} < 2\left(1 + 2C_0\varepsilon^2 \log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right)\right)$$

for all $t \in \left[1, \widetilde{T}\right], |\xi| \le 1$. Multiplying both sides of equation (4.3) by $\omega$ we see that

$$\partial_t \omega^2 = -2C_0 t^{-1} \left\langle \widetilde{\xi} \right\rangle^{-\frac{\alpha}{2}} r^2 \omega \sin 2\omega + 2\omega r^{-1} \operatorname{Im} h e^{-i\omega} \le \omega^2 \partial_t \log r^2 + Cr^{-1}|h|,$$

since $2\omega \sin 2\omega \ge \frac{8}{\pi}\omega^2$ for $|\omega| \le \frac{\pi}{4}$. Then integration with respect to time yields

$$|\omega(t)|^2 \le r^2 \left(\frac{\left|\arg y_1\right|^2}{|y_1|^2} + C\int_1^t r^{-3}|h|\, d\tau\right) \le \left(2\left|\arg y_1\right|^2 + C\varepsilon^{2-7\gamma}\right)\left(1 + C_0\delta^2 \log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right)\right)^{-\frac{1}{4}}$$

for all $t \in \left[1, \widetilde{T}\right], |\xi| \le 1$. Hence $|\omega(t,\xi)| < \frac{\pi}{4}$. We get a contradiction, therefore the estimates of the lemma are true for all $t \in [1, T]$. Lemma 4.2 is proved. □

In the next lemma we obtain the estimates of the function $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$ in the norm $\mathbf{Z}_T$.

**Lemma 4.3.** *Let the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$ with a norm $\|u_0\|_{\mathbf{H}^1} + \|u_0\|_{\mathbf{H}^{0,1}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Also suppose that condition* (1.2) *is fulfilled. Assume that $\sup_{t\in[0,T]} \frac{1}{K(t)} \left\|\widehat{\varphi}_\xi(t)\right\|_{\mathbf{L}^2} \le C\varepsilon$. Then the estimate $\|\widehat{\varphi}\|_{\mathbf{Z}_T} < C\varepsilon$ is true.*

Proof. We argue by a contradiction. By the continuity we can find a maximal time $\widetilde{T} \in (1, T]$ such that $\|\widehat{\varphi}\|_{\mathbf{Z}_{\widetilde{T}}} \le C\varepsilon$. Hence we have $\|\widehat{\varphi}\|_{\mathbf{X}_{\widetilde{T}}} \le C\varepsilon$. By Lemma 4.1 we get (4.1). Applying the result of Lemma 4.2 we get $|y(t,\xi)| \le 2\varepsilon\left(1 + C_0\delta^2 \log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right)\right)^{-\frac{1}{2}}$. Also as in the proof of Lemma 4.1 we obtain $\widehat{\varphi} = y + O\left(\varepsilon^3 Q^{-\frac{3}{2}}\right)$. Therefore

$$\left|\widehat{\varphi}(t,\xi)\right| < 3\varepsilon\left(1 + \delta_1 \log\left(t\left\langle \widetilde{\xi} \right\rangle^{-2}\right)\right)^{-\frac{1}{2}}$$

Also note that if $t\xi^2 \leq t^{\frac{1}{2}}$, then $\left(1+\delta_1 \log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{-\frac{1}{2}} \leq \sqrt{2}Q^{-\frac{1}{2}}(t)$ and if $t\xi^2 > t^{\frac{1}{2}}$, then $\left\langle\widetilde{\xi}\right\rangle^{-\gamma} \leq \sqrt{2}Q^{-\frac{1}{2}}(t)$. Thus we have

$$\left\|\left\langle\widetilde{\xi}\right\rangle^{-\gamma}\varphi(t)\right\|_{\mathbf{L}^\infty} < 3\varepsilon \sup_{\xi\in\mathbb{R}}\left\langle\widetilde{\xi}\right\rangle^{-\gamma}\left(1+\delta_1\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{-\frac{1}{2}} \leq 5\varepsilon Q^{-\frac{1}{2}}(t).$$

Hence $\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_{\widetilde{T}}} \leq C\varepsilon$. We obtain a contradiction. Lemma 4.3 is proved. □

## 5. Proof of Theorem 1.1

Lemma 3.2 shows that the a priori estimate of $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u$ in the norm $\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_T}$ implies the a priori estimate of the norm $\sup_{t\in[0,T]}\frac{1}{K(t)}\left\|\widehat{\varphi}_\xi(t)\right\|_{\mathbf{L}^2}$. On the other hand by Lemma 4.3 it is shown that the a priori estimate of $\sup_{t\in[0,T]}\frac{1}{K(t)}\left\|\widehat{\varphi}_\xi(t)\right\|_{\mathbf{L}^2}$ implies the a priori estimate of $\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_T}$. Therefore the global existence of solution $u \in \mathbf{C}\left([1,\infty);\mathbf{H}^1\cap\mathbf{H}^{0,1}\right)$ to the Cauchy problem (1.1) satisfying a priori estimate $\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon$ follows by a standard continuation argument from Lemma 3.2, Lemma 4.3 and the local existence Theorem 3.1.

We now prove the asymptotic formula (1.3). By the Factorization Technique we have $u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}M\mathcal{Q}\widehat{\varphi}$. Then applying Lemma 2.2 we obtain

$$u(t) = \mathcal{D}_t\mathcal{B}M\mathcal{Q}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}M\frac{1}{\sqrt{i\Lambda''(\xi)}}\widehat{\varphi}(t,\xi) + O\left(t^{-\frac{1}{2}}(\log t)^{-\frac{3}{2}}\right).$$

Then as in the proof of Lemma 4.1 we find

$$\widehat{\varphi} = y + O\left(\varepsilon^3 Q^{-\frac{3}{2}}\right) = re^{i\omega} + O\left(\varepsilon^3 Q^{-\frac{3}{2}}\right)$$

As in the proof of Lemma 4.2 we obtain

$$|\omega| \leq \frac{\pi}{4}\left(1 + C_0\delta^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{-\frac{1}{8}}$$

and

$$\begin{aligned} r(t) &= \left|\widehat{u_0}(\xi)\right|\left(1 + C_0\left|\widehat{u_0}(\xi)\right|^2\int_1^t\left\langle\sqrt{\tau}\xi\right\rangle^{-\frac{\alpha}{2}}\cos 2\omega\frac{d\tau}{\tau} + O\left(\left(\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{\frac{3}{4}}\right)\right)^{-\frac{1}{2}} \\ &= \left|\widehat{u_0}(\xi)\right|\left(1 + \left|\widehat{u_0}(\xi)\right|^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{-\frac{1}{2}} + O\left(\left(1 + \left|\widehat{u_0}(\xi)\right|^2\log\left(t\left\langle\widetilde{\xi}\right\rangle^{-2}\right)\right)^{-\frac{3}{4}}\right). \end{aligned}$$

Therefore we get

$$u(t) = \mathcal{D}_t\mathcal{B}M\left(\frac{\left|\widehat{u_0}(\xi)\right|}{\sqrt{i\Lambda''(\xi)}}\left(1 + \left|\widehat{u_0}(\xi)\right|^2\log\frac{t}{1+t\xi^2}\right)^{-\frac{1}{2}} + O\left(t^{-\frac{1}{2}}\left(\log\frac{t}{1+t\xi^2}\right)^{-\frac{3}{4}}\right)\right)$$

which implies the asymptotics (1.3). Theorem 1.1 is proved.

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