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SHARP GROWTH ESTIMATES FOR WARPING FUNCTIONS IN MULTIPLY WARPED PRODUCT MANIFOLDS

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Abstract. By applying an average method in PDE, we obtain a dichotomy between "constancy" and "infinity" of the warping functions on complete noncompact Riemannian manifolds for an appropriate isometric immersion of a multiply warped product manifold $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian manifold.

Generalizing the earlier work of the authors in [9], we establish sharp inequalities between the mean curvature of the immersion and the sectional curvatures of the ambient manifold under the influence of quantities of a purely analytic nature (the growth of the warping functions). Several applications of our growth estimates are also presented.

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Contents

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1. Introduction

Warped products play very important roles in Differential Geometry and Physics. Examples of warped product include Riemannian manifolds of constant curvature and the best relativistic model of the Schwarzschild space-time that describes the out space around a mass star or a black hole.

In [9], B.-Y. Chen and S. Wei obtained the following necessary condition for an arbitrary isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian m-manifold $\tilde{M}^m_c$ with sectional curvatures bounded from above by a constant c, generalizing the work of B.-Y. Chen in [3] on warped product submanifolds in a Riemannian manifold $R^m(c)$ of constant sectional curvature c.

Theorem A ([9, Theorem 3.1]). *For any isometric immersion $\phi : N_1 \times_f N_2 \to \tilde{M}^m_c$ from a warped product $N_1 \times_f N_2$ into a Riemannian m-manifold $\tilde{M}^m_c$ with sectional curvatures bounded from above by a constant c, the warping function f satisfies*

$$-\frac{(n_1+n_2)^2}{4n_2}H^2 - n_1 c \le \frac{\Delta f}{f} \tag{1}$$

where $n_1 = \dim N_1$ and $n_2 = \dim N_2$, $H^2 = \langle H, H\rangle$ is the squared mean curvature of ϕ, and Δf is the Laplacian of f on N_1 (defined as the divergence of the gradient vector field of f, cf. (10)).

The equality sign (1) *holds if and only if ϕ is a mixed totally geodesic immersion with* $\operatorname{trace} h_1 = \operatorname{trace} h_2$*, where h_1 and h_2 are the restriction of the second fundamental form h of ϕ restricted to N_1 and N_2, respectively, and at each point $p = (p_1, p_2) \in N$, c satisfies $c = K(u, v) = \max K(p)$, for every unit vector $u \in T^1_{p_1}N_1$ and every unit vector $v \in T^1_{p_2}(N_2)$.*

On the other hand, the second author extended in [15] the scope of L^q or q-integrable functions on complete noncompact Riemannian manifolds to functions with "*p-balanced*" growth depending on q, and introduced the concepts of their counter-part to "*p-imbalanced*" growth (cf. Definition 9). By coupling these growth estimates with the above inequality (1), Chen and Wei establish in [9] some sharp inequalities between quantities of a geometric nature (the mean curvature of the immersion, the sectional curvatures of the ambient manifold) and quantities of a purely analytic nature (the growth of the warping function).

Theorem B ([9]). *If f is nonconstant and* two-balanced *for some $q > 1$, then for every Riemannian n_2-manifold N_2 and every isometric immersion ϕ of the warped product $N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}^m_c$ with $c \le 0$, the mean* *curvature H of ϕ satisfies*

$$H^2 > \frac{4n_1 n_2 |c|}{(n_1 + n_2)^2} \tag{2}$$

at some points.

Hence we immediately find a dichotomy between “constancy” and “infinity” (two-imbalanced) of the warping functions on complete noncompact Riemannian manifolds for an appropriate isometric immersion:

Corollary A ([9]). *Suppose the squared mean curvature of the isometric immersion $\phi : N_1 \times_f N_2 \to \tilde{M}_c^m$ satisfies*

$$H^2 \leq \frac{4n_1 n_2 |c|}{(n_1 + n_2)^2} \tag{3}$$

everywhere on $N_1 \times_f N_2$. Then the warping function f is either a constant or it has two-imbalanced *growth for every $q > 1$.*

Applications of these new inequalities are also presented, among which there are some results on the nonexistence of isometric minimal immersions between certain types of Riemannian manifolds:

Theorem C ([9]). *Suppose $q > 1$ and the warping function f is* two-balanced*. If N_2 is compact, then there does not exists an isometric minimal immersion from $N_1 \times_f N_2$ into any Euclidean space.*

A Riemannian manifold is said to be *negatively curved* (respectively, *non-positively curved*) if it has negative (respectively, *non-positively curved*) sectional curvatures.

Corollary B ([9]). *If f is an L^q function on N_1 for some $q > 1$, then for any Riemannian manifold N_2 the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into any non-positively curved Riemannian manifold.*

For further extension, let $N = N_1 \times \cdots \times N_k$ denote the Cartesian product of k Riemannian manifolds $(N_1, g_1) \cdots, (N_k, g_k)$, and $\pi_i : N \to N_i$ be the canonical projection of N onto N_i , $1 \leq i \leq k$. If $f_2, \cdots f_k : N_1 \to \mathbb{R}^+$ are smooth positive-valued functions, then

$$g = \pi_1^* g_1 + \sum_{i=2}^{k} (f_i \circ \pi_1)^2 \pi_i^* g_i$$

defines a Riemannian metric on N , called multiply warped product metric. The product manifold N endowed with g is denoted by $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$.

Denote by $\operatorname{trace} h_i$ the trace of the second fundamental form h of $N = N_1 \times \cdots \times N_k$ into a Riemannian manifold restricted to N_i .

B.-Y. Chen and F. Dillen proved in [6] the following.

Theorem D. *Let $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to M$ be an isometric immersion of a multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into an arbitrary Riemannian manifold M. Then we have*

$$-\frac{n^2(k-1)}{2k} H^2 - n_1(n-n_1) \max \tilde{K} \leq \sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \tag{4}$$

where $n = \sum_{i=1}^{k} n_i$ and $\max \tilde{K}(p)$ denotes the maximum of the sectional curvature function of the ambient space M restricted to two-plane sections of the tangent space T_pN of N at $p = (p_1, \ldots, p_k)$.

The equality sign of (4) *holds identically if and only if the following two conditions hold*

i) *ϕ is mixed totally geodesic such that* $\operatorname{trace} h_1 = \cdots = \operatorname{trace} h_k$

ii) *At each point $p \in N$, the sectional curvature function $\tilde{K}$ satisfies $\tilde{K}(u,v) = \max \tilde{K}(p)$ for every $u \in T^1_{p_1} N_1$ and every $v \in T^1_{p_2, \ldots p_k}(N_2 \times \cdots N_k)$.*

One main purpose of this article is to prove the following theorem which extends Theorem B, in particular, inequality (2) to arbitrary isometric immersions of multiply warped product manifolds into an arbitrary Riemannian manifold.

Theorem 1. *If for each j, $2 \leq j \leq k$, f_j is nonconstant and* two-balanced *with some $q_j > 1$, then, for any multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ in a Riemannian manifold M, the mean curvature H of N in M satisfies*

$$H^2 > \frac{-2kn_1(n-n_1)}{n^2(k-1)} \max \tilde{K} \tag{5}$$

at some points, where $\max \tilde{K}$ is defined in Theorem D.

In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then (5) *holds at some points.*

In particular, if M is a Riemannian manifold of constant sectional curvature $c \leq 0$, then Theorem 1 reduces to the following.

Theorem 2. *If for each j, $2 \leq j \leq k$, f_j is nonconstant and* two-balanced *with some $q_j > 1$, then, for any multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$*

in a Riemannian manifold $R^m(c)$ of constant sectional curvature $c \leq 0$, the mean curvature H of N in $R^m(c)$ satisfies

$$H^2 > \frac{2kn_1(n-n_1)}{n^2(k-1)}c \tag{6}$$

at some points.

In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then (6) *holds at some points.*

Theorems 1 and 2 are sharp and inequalities (5) and (6) are optimal. For details, we refer to Remark 18, Example 3.1, Example 3.2, and Remark 19.

In views of Theorem 1, we give the following dichotomy.

Theorem 3. *Suppose the squared mean curvature of the isometric immersion of a multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian manifold satisfies*

$$H^2 \leq \frac{-2kn_1(n-n_1)}{n^2(k-1)} \max \tilde{K} \tag{7}$$

everywhere on N. Then there exists an integer $i, 2 \leq i \leq k$ such that either the warping function f_i is a constant or f_i has two-imbalanced *growth for every $q_i > 1$.*

Some other applications of Theorem 1 are the following.

Corollary 4. *If for each $j, 2 \leq j \leq k$, f_j is nonconstant and* two-balanced *for some $q_j > 1$, then there does not exist a minimal immersion of any multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian manifold whose maximum sectional curvature is nonpositive.*

In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then there does not exist a minimal immersion of any multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Euclidean space.

Applying the growth estimates in Theorem 15 and the average method in PDE in Proposition 16, we have the following Liouville property and characterization results.

Corollary 5. *Suppose the squared mean curvature of the isometric immersion ϕ of a multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a complete, simply-connected Riemannian manifold $R^m(c)$ of constant sectional curvature c satisfies* (7) *everywhere on N. If for each $j, 2 \leq j \leq k$, f_j is* two-balanced *for some $q_j > 1$, then we have*:

1) *Every warping function* $f_j, 2 \leq j \leq k$ *is constant.*

2) *The isometric immersion* ϕ *is a minimal immersion into a Euclidean space.*

3) *The isometric immersion* ϕ *is a warped product immersion.*

Corollary 6. *Let each* f_j , $2 \leq j \leq k$ *be* two-balanced *for some* $q_j > 1$. *Then we have*:

1) *Every multiply warped product* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *does not admit an isometrically minimal immersion into any Riemannian manifold of negative sectional curvature.*

2) *If* N_k *is compact, then* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *does not admit an isometrically minimal immersion into a Euclidean space.*

We state a special case of Corollary 6 as the following.

Corollary 7. *If each* f_j , $2 \leq j \leq k$, *is in* L^{q_j} *for some* $q_j > 1$, *then we have*

1) *Every multiply warped product* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *does not admit an isometrically minimal immersion into any negatively curved Riemannian manifold.*

2) *If* N_k *is compact, then* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *does not admit an isometrically minimal immersion into a Euclidean space.*

A map

$$\psi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to M_1 \times_{\rho_2} M_2 \times \cdots \times_{\rho_k} M_k$$

between two multiply warped product manifolds $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ and $M_1 \times_{\psi_2} M_2 \times \cdots \times_{\psi_k} M_k$ is said to be a *warped product immersion* if ψ is given by $\psi(x_1, \cdots, x_k) = (\psi_1(x_1), \cdots, \psi_k(x_k))$ is an isometric immersion, where $\psi_i : N_i \to M_i, i = 2, \cdots, k$ are isometric immersions, and $f_i = \rho_i \circ \psi_1 : N_1 \to \mathbb{R}^+$ for $i = 2, \cdots, k$.

By applying Theorem D, Proposition 16 and Theorem E (Nölker's Theorem), we have

Corollary 8. *If for each* $j, 2 \leq j \leq k$, f_j *is* two-balanced *for some* $q_j > 1$, *then every isometric minimal immersion of a multiply warped product* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *into a Euclidean space is a warped product immersion.*

The technique used in this article is to apply the Average Method in PDE in Proposition 16 and the Growth Estimates in Theorem 15 to study multiply warped products. In contrast to *an extrinsic average variational method in the calculus of variations* [10,16,17], where the sum of analytic quantities is strictly negative, *the average method in* PDE in this article deals with the nonnegative sum of analytic quantities (cf. Remark 2.1).

The techniques used in this article are sufficient general to apply to multiply warped product manifolds totally real isometrically immersed into complex space forms, as well as into quaternionic space forms. We also use the same technique for multiply warped product manifolds to treat doubly warped product manifolds in the last section.

2. Preliminaries

Let N be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ in $\tilde{M}^m$ such that, restricted to N, the vectors $e_1, \dots, e_n$ are tangent to N and $e_{n+1}, \dots, e_m$ are normal to N.

For a submanifold N in $\tilde{M}^m$, let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of N and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are then given respectively by (see, for instance, [4,5])

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y) \\ \tilde{\nabla}_X \xi &= -A_\xi X + D_X \xi \end{aligned} \tag{8}$$

for vector fields X, Y tangent to N and ξ normal to N, where h is the second fundamental form, D the normal connection, and A the shape operator of the submanifold. Let $\{h_{ij}^r\}$, $i, j = 1, \dots, n$; $r = n+1, \dots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \dots, e_n, e_{n+1}, \dots, e_m$.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n} \operatorname{trace} h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \tag{9}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of the tangent bundle TN of N. The squared mean curvature is given by

$$H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$$

where $\langle\ ,\ \rangle$ denotes the inner product. A submanifold N is called *minimal* in $\tilde{M}^m$ if its mean curvature vector vanishes identically.

Let P be a Riemannian k-manifold and $\{e_1, \ldots, e_k\}$ be an orthonormal frame field on P. For a differentiable function φ on P, the *Laplacian* of φ is defined by the divergence of the gradient of φ, or the trace of the Hessian φ, i.e.,

$$\Delta\varphi = \sum_{j=1}^{k} \{e_j e_j \varphi - (\nabla_{e_j} e_j)\varphi\}. \tag{10}$$

A function φ on P is said to be *harmonic* (respectively *subharmonic* or *superharmonic*) if we have $\Delta\varphi = 0$ (respectively $\Delta\varphi \geq 0$ or $\Delta\varphi \leq 0$) on P.

An isometric immersion

$$\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to M$$

of a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Riemannian m-manifold M is called *mixed totally geodesic* if its second fundamental form h satisfies $h(\mathcal{D}_i, \mathcal{D}_j) = 0$ for any distinct $i, j \in \{1, \cdots, k\}$, where $\mathcal{D}_i$ denotes the distribution obtained from the vectors tangent to the horizontal lifts of N_i .

We recall the following results for later use.

Theorem E ([13, Nölker's Theorem]). *Let $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to R^m(c)$ be an isometric immersion into a Riemannian manifold $R^m(c)$ of constant sectional curvature c . If ϕ is mixed totally geodesic, then locally ϕ is a warped product immersion*

In the following, let us assume that N_1 is a noncompact complete Riemannian manifold and $B(x_0; r)$ denotes the geodesic ball of radius r centered at $x_0 \in N_1$.

We recall some notions from [15].

Definition 9. *A function on N_1 is said to have* p-balanced growth (*or, simply,* is p-balanced) *if it is one of the following: p-finite, p-mild, p-obtuse, p-moderate, and p-small; it has* p-imbalanced growth, *or simply is* p-imbalanced *otherwise.*

Notice that the definitions of "*p-finite*, *p-mild*, *p-obtuse*, *p-moderate*, *p-small*" and their counter-parts "*p-infinite*, *p-severe*, *p-acute*, *p-immoderate*, *p-large*" growth depend on q, and q will be specified in the context in which the definition is used.

We have discussed their definitions in [9, Definition 4.1-4.5]. For completeness we include them as follows (please see also [15]).

Definition 10. *A function f on N_1 is said to have p*-finite growth (*or, simply,* is p-finite) *if there exists $x_0 \in N_1$ such that*

$$\lim_{r\to\infty} \inf \frac{1}{r^p} \int_{B(x_0;r)} |f|^q \mathrm{d}v < \infty \tag{11}$$

it has p-infinite growth (*or, simply,* is p-infinite) *otherwise.*

Definition 11. *A function f has* p-mild growth (or, simply, is p-mild) if there exists $x_0 \in N_1$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have

$$\sum_{j=\ell_0}^{\infty} \left(\frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q \mathrm{d}v} \right)^{\frac{1}{p-1}} = \infty \tag{12}$$

and has p-severe growth (or, simply, is p-severe) otherwise.

Definition 12. *A function f has p*-obtuse growth (*or, simply,* is p-obtuse) *if there exists $x_0 \in N_1$ such that for every $a > 0$, we have*

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q \mathrm{d}v} \right)^{\frac{1}{p-1}} \mathrm{d}r = \infty \tag{13}$$

and has p-acute growth (*or, simply,* is p-acute) *otherwise.*

Definition 13. *A function f has p*-moderate growth (*or, simply,* is p-moderate) *if there exist $x_0 \in N_1$, and $F(r) \in \mathcal{F}$, such that*

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q \mathrm{d}v < \infty. \tag{14}$$

And it has p-immoderate growth (*or, simply,* is p-immoderate) *otherwise, where*

$$\mathcal{F} = \left\{ F : [a, \infty) \longrightarrow (0, \infty);\ \int_a^\infty \frac{\mathrm{d}r}{rF(r)} = +\infty \ \text{ for some } a \geq 0 \right\}. \tag{15}$$

(Notice that the functions in $\mathcal{F}$ are not necessarily monotone.)

Definition 14. *A function f has p*-small growth (*or, simply,* is p-small) *if there exists $x_0 \in N_1$, such that for every $a > 0$, we have*

$$\int_a^\infty \left(\frac{r}{\int_{B(x_0;r)} |f|^q \mathrm{d}v} \right)^{\frac{1}{p-1}} \mathrm{d}r = \infty \tag{16}$$

and has p-large growth (*or, simply,* is p-large) *otherwise.*

We recall the following result from [9] for later use.

Theorem 15 (Warping Function Growth Estimates) *Let N_1 be a noncompact complete Riemannian manifold and $f : N_1 \to \mathbb{R}^+$ be a C^2 positive function satisfying $\Delta f/f \geq 0$ on N_1. Then either f is constant or f is two-imbalanced for every $q > 1$.*

Proof: Follow exactly the proof of Theorems 4.1, 4.2, 4.3, 4.4 and 4.5 in [9, pp.586-590] and use Definition 9, the assertion follows. ■

We also need the following result.

Proposition 16 (An Average Method in PDE) *Let $c_2, \cdots, c_k$ be $k-1$ positive constants and let $f_2, \cdots, f_k$ be positive-valued functions defined on a complete noncompact manifold N_1 such that $\sum_{j=2}^k c_j \Delta f_j / f_j \geq 0$, Then we have*:

1) *There exists an integer i, $2 \leq i \leq k$, such that either f_i is a constant or f_i is two-imbalanced for every $q_i > 1$.*
2) *If each f_j, $2 \leq j \leq k$, is two-balanced for some $q_j > 1$, then all of $f_2, \cdots, f_k$ are constant functions.*

Proof: If $\sum_{j=2}^k c_j \Delta f_j / f_j \geq 0$ holds, then there exists at least i, $2 \leq i \leq k$, such that $\Delta f_i / f_i \geq 0$ holds. Or $\sum_{j=2}^k c_j \Delta f_j / f_j < 0$, contradicting to the hypothesis. Therefore statement (1) of this proposition follows from Theorem 15.

For statement (2), it follows from the assumptions that f_i is two-balanced. Hence statement (1) implies that f_i is constant. So we have $\Delta f_i = 0$ and

$$\sum_{j \neq i} c_j \frac{\Delta f_j}{f_j} = \sum_{j=2}^{k} c_j \frac{\Delta f_j}{f_j} \geq 0.$$

Now, by applying statement 1) again to $\sum_{j \neq i} c_j \Delta f_j / f_j \geq 0$, we can find the second constant warping function $f_{i'}$ such that $\sum_{j \neq i, i'} c_j \Delta f_j / f_j \geq 0$. Now, using the same method iteratively, we conclude that all of $f_2, \cdots, f_k$ are all constant. This proves statement 2). ■

Remark 17. The average method given in Proposition 16 is in contrast to *an extrinsic average variational method in the calculus of variations* [10, 16, 17], where the sum of analytic quantities, the second variation formulas of functionals such as the mass, p-energy, or Yang-Mills functional (over a set of distinguished variation vector fields) is *strictly negative*. Our average method in PDE in Proposition 16 deals with the *nonnegative* sum of analytic quantities, the Laplacian of warping functions.

3. Proof of Theorem 1, Theorem 3 and Corollaries 4 - 8

The proofs of these results are based on Theorem D via the Average Method in PDE given in Proposition 16 and the Warping Function Growth Estimates given in Theorem 15.

Proof of Theorem 1. Suppose contrary to (5), i.e., there were an isometric immersion ϕ whose mean curvature H satisfying

$$H^2 \le \frac{-2kn_1(n-n_1)\max \tilde{K}}{n^2(k-1)} \tag{17}$$

everywhere on N . This would imply by multiplying both sides of (7) by a positive number $n^2(k-1)/2k$, or equivalently

$$0 \le -\frac{n^2(k-1)}{2k}H^2 - n_1(n-n_1)\max \tilde{K}.$$

On the other hand, Theorem D would imply

$$-\frac{n^2(k-1)}{2k}H^2 - n_1(n-n_1)\max \tilde{K} \le \sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j}.$$

After combining this with (17) or its equivalent inequality, we find

$$\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \ge -\frac{n^2(k-1)}{2k}H^2 - n_1(n-n_1)\max \tilde{K} \ge 0. \tag{18}$$

Now, after applying the Average Method in PDE stated in Proposition 16(1), we would conclude from (18) that some f_i, $2 \le i \le k$, could be constant *or* f_i would be two-imbalanced for every $q_i > 1$, contradicting the assumption that f_i is nonconstant *and* two-balanced for some $q_i > 1$. Indeed, "f_i would be constant" contradicts "f_i is nonconstant" and "f_i would be two-imbalanced for every $q_i > 1$." contradicts "f_i is two-balanced for some $q_i > 1$".

To prove the last assertion, we observed that every L^q function has *two-finite*, *two-mild*, *two-obtuse*, *two-moderate*, *two-small* growth for the same q (cf. [18, Proposition 2.3]). For example, if f defined on N_1 is in L^q, then f is two-finite with respect to the same q. Indeed, there exists $x_0 \in N_1$ such that (11), where $p = 2$ holds:

$$\begin{aligned}
\lim_{r\to\infty} \inf \frac{1}{r^2}\int_{B(x_0;r)} |f|^q \mathrm{d}v &\le \lim_{r\to\infty} \inf \frac{1}{r^2}\int_{N_1} |f|^q \mathrm{d}v \\
&= \lim_{r\to\infty} \inf \frac{1}{r^2}\, C, \text{ for some constant } C > 0 \\
&= 0 < \infty.
\end{aligned}$$

Definition 9 and the first assertion of Theorems 1 complete the proof. ■

Proof of Theorem 3. Let $\phi : N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to M$ be an isometric immersion of a multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ in a Riemannian manifold M. If the mean curvature H of N in M satisfies (7) on N, then it follows from the inequality (4) of Theorem D and (7) that $\sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} \geq 0$. Hence, after applying the Average Method in PDE stated in Proposition 16(1), we conclude that some f_i, $2 \leq i \leq k$, could be constant *or* f_i would be two-imbalanced. ■

Proof of Corollary 4. Suppose contrary, such an immersion would violate (5) and hence contradicts Theorem 1. ■

Proof of Corollary 5. Statement $1)$ of Corollary 5 follows from Theorem 3 and Proposition 16(2).

In view of (7) and Proposition 16 2), we have

$$0 \leq -H^2 - \frac{2kn_1(n-n_1)c}{n^2(k-1)} \leq \sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} = 0.$$

Therefore we obtain $H = c = 0$, which implies Statement 2).

Statement 3) follows immediately from Corollary 8. ■

Proof of Corollary 6. To prove Statement 1), let us suppose contrary. Then it follows from Theorem D that

$$0 < -H^2 - \frac{2kn_1(n-n_1)\max \tilde{K}}{n^2(k-1)} \leq \sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j}. \tag{19}$$

Now, by Theorem 15 1), (19) implies the constancy of f_i for some $2 \leq i \leq k$. Thus

$$0 < \sum_{j \neq i} \frac{\Delta f_j}{f_j}. \tag{20}$$

Therefore, after applying Proposition 16(2) to (20) we obtain the constancy of $f_2, \cdots, f_k$, which leads to $0 < 0$, a contradiction.

For statement 2), let us suppose contrary. Then inequality (18) would be true. Hence by Proposition 16 1), we would have the constancy of f_i for some $2 \leq i \leq k$. Thus

$$0 \leq \sum_{j \neq i} n_j \frac{\Delta f_j}{f_j}. \tag{21}$$

Now, applying Proposition 16 2) shows the constancy of $f_2, \cdots, f_k$. So, it follows from Theorem D that ϕ is mixed totally geodesic and hence, by Moore's lemma [12], we conclude that

$$\phi = (\phi_1, \cdots, \phi_k) : N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to \mathbb{E}^{m_1} \times \cdots \times \mathbb{E}^{m_k} = \mathbb{E}^m$$

is a product minimal immersion, which contradicts to the fact that there is no compact minimal submanifold N_k in the Euclidean space $\mathbb{E}^{m_k}$. ■

Proof of Corollary 7. Follows at once from Corollary 6 and the fact that every L^q function with $q > 1$ on N_1 is two-balanced for the same $q > 1$ on N_1. ■

Proof of Corollary 8. In view of Theorem D and $H = c = 0$, we have

$$0 = -H^2 - \frac{2kn_1(n-n_1)c}{n^2(k-1)} = \sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j} = 0.$$

Now assertion follows from Theorem E [13, Nölker's Theorem]. ■

Remark 18. *The following two examples show that* Theorem 1 *is false if either f_j is constant or f_j is two-imbalance for every $q_j > 1$.*

Example 3.1. Let $N_1, \ldots, N_k$ be k copies of the real line $\mathbf{R}$ and let us put $f_j = 1$ for $j = 2, \ldots, k$. Then $N = N_1 \times_1 N_2 \times \cdots \times_1 N_k$ is the Euclidean k-space $\mathbb{E}^k$. Clearly, for a totally geodesic immersion of N into $\mathbb{E}^{k+1}$, inequality (5) is false.

Example 3.2. Let $N_1 = \{x \in \mathbf{R};\, x > 0\}$ and $N_2, \ldots, N_k$ be $k-1$ copies of $\mathbf{R}$. If we put $f_2 = \cdots = f_k = x$, then each f_j is two-imbalance for every $q_j > 1$ and $N = N_1 \times_x N_2 \times \cdots \times_x N_k$ is an open subset of $\mathbb{E}^k$. Again, for a totally geodesic immersion of N into $\mathbb{E}^{k+1}$, inequality (5) is false.

Remark 19. Theorems 1 *and* 2 *are sharp in the sense that inequality* (5) *and* (6) *are false if either f_j were constant or f_j were two-imbalanced for every $q_j > 1$* (*For details, we refer to* Remark 18*, and* Examples 3.1–3.2 *above*)*. Furthermore,* Theorem 3 *shows that inequality* (5) (*respectively,* (6)) *is best possible for* Theorem 1 (*respectively, for* Theorem 2).

4. Multiply Warped Product Manifolds into Complex or Quaternionic Space Forms

A submanifold N of a Kähler manifold M is said to be *totally real* if the almost complex structure J of M carries each tangent space of N into its corresponding normal space (cf. [4, 8]). Similarly, one has the notion of totally real submanifolds in quaternionic Kähler manifolds (cf. [7]).

B.-Y. Chen and F. Dillen proved

Theorem F ([6]). *Let $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k \to \tilde{M}^m(4c)$ be a totally real isometric immersion of the multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a complex space form of constant holomorphic sectional curvature $4c$ or in a quaternionic space form of constant quaternionic sectional curvature $4c$. Then*

$$-\frac{n^2}{4}H^2 - n_1(n-n_1)c \leq \sum_{j=2}^{k} n_j \frac{\Delta f_j}{f_j}, \qquad n = \sum_{i=1}^{k} n_i. \tag{22}$$

By applying the same techniques, i.e., Theorem 15 (Warping Functions Growth Estimates) and Proposition 16 (An Average Method in PDE), we also have the following results.

Theorem 20. *If for each $j, 2 \leq j \leq k$, f_j is nonconstant and* two-balanced *for some $q_j > 1$, then for any multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ totally real isometrically immersed in a complex space form of constant holomorphic sectional curvature $4c$ or in a quaternionic space form of constant quaternionic sectional curvature $4c$, the mean curvature H of ϕ satisfies*

$$H^2 > \frac{-4n_1(n-n_1)c}{n^2} \tag{23}$$

at some points.

In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then (23) *holds at some points.*

As applications of Theorem 20 , we have the following.

Corollary 21. *If for each $j, 2 \leq j \leq k$, f_j is nonconstant and* two-balanced *for some $q_j > 1$, then there does not exist a totally real minimal immersion of any multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a complex space form of constant holomorphic sectional curvature $4c \leq 0$ or into a quaternionic space form of constant quaternionic sectional curvature $4c \leq 0$.*

In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then there does not exist an isometric minimal immersion of $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into $\tilde{M}^m(0)$.

Another application of Theorem 20 is the following dichotomy.

Corollary 22. *Suppose the squared mean curvature of an isometric immersion of a multiply warped product* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *into a Riemannian manifold* $R^m(c)$ *of constant sectional curvature* c *satisfies*

$$H^2 \le \frac{-4n_1(n-n_1)c}{n^2} \tag{24}$$

everywhere on N. *Then there exists an integer* $i, 2 \le i \le k$, *such that the warping function* f_i *is either a constant or for every* $q_i > 1$, f_i *has* two-imbalanced *growth.*

By applying the Growth Estimates in Theorem 15 and an Average Method in PDE in Proposition 16, we have the following.

Corollary 23. *Suppose the squared mean curvature of a totally real isometric immersion* ϕ *of a multiply warped product* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *into a complex space form of constant holomorphic sectional curvature* $4c$ *or a quaternionic space form of constant quaternionic sectional curvature* $4c$ *satisfies* (24) *everywhere on* N. *If for each* $j, 2 \le j \le k$, f_j *is* two-balanced *for some* $q_j > 1$, *then*

1) *Every warping function* $f_j, 2 \le j \le k$, *is constant.*

2) *The isometric immersion* ϕ *is a minimal immersion into* $\tilde{M}^m(0)$.

Corollary 24. *Let each* f_j , $2 \le j \le k$, *be* two-balanced *for some* $q_j > 1$. *Then*

1) *Every multiply warped product* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *does not admit an isometrically totally real minimal immersion into any complex space form of negative constant holomorphic sectional curvature* $4c$ *or a quaternionic space form of negative constant quaternionic sectional curvature* $4c$.

2) *If* N_1 *is compact, then* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *does not admit an isometrically totally real minimal immersion into* $\tilde{M}^m(0)$.

We state a special case of Corollary 24 as follows.

Corollary 25. *If each* f_j , $2 \le j \le k$, *is in* L^{q_j} *for some* $q_j > 1$, *then we have*:

1) *Every multiply warped product* $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ *does not admit an isometrically totally real minimal immersion into a complex space form of negative constant holomorphic sectional curvature* $4c$ *or into a quaternionic space form of negative constant quaternionic sectional curvature* $4c$.

2) *If N_1 is compact, then $N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ does not admit an isometrically totally real minimal immersion into $\tilde{M}^m(0)$.*

Since the proofs of these results can be done in the same way as in Section 3, we omit their proofs.

5. Doubly Warped Products

Doubly warped products are natural generalization of (ordinary) warped products.

Definition 26. *A doubly warped product of Riemannian manifolds (N_1, g_1) and (N_2, g_2) is a product manifold ${}_{f_2}N_1 \times_{f_1} N_2$ equipped with metric $g = f_2^2 g_1 \oplus f_1^2 g_2$, where $f_1 : N_1 \to \mathbb{R}^+$ and $f_2 : N_2 \to \mathbb{R}^+$ are positive-valued smooth functions.*

As an extension of Theorem A from [9], A. Olteanu proved the following.

Theorem G ([14]). *Let $\phi : {}_{f_2}N_1 \times_{f_1} N_2 \to M$ be an isometric immersion of a doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ into an arbitrary Riemannian manifold M. We have*

$$-\frac{(n_1+n_2)^2}{4} H^2 - n_1 n_2 \max \tilde{K} \leq n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \tag{25}$$

where $n_i = \dim N_i$ and Δ_i is the Laplacian of N_i , for $i = 1, 2$.

The equality sign holds identically if and only if the following conditions hold:

i) *ϕ is mixed totally geodesic such that* $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

ii) *At each point $x = (x_1, x_2) \in N$, $\tilde{K}$ satisfies $\tilde{K}(u, v) = \max K(x)$ for each unit vector $u \in T^1_{x_1} N_1$ and every $v \in T^1_{x_2} N_2$.*

Similarly, by applying the same techniques via Theorem 15 (Warping Functions Growth Estimates) and Proposition 16 (An Average Method in PDE), we also have the following.

Theorem 27. *If f_1, f_2 are nonconstant and* two-balanced *for some $q_1, q_2 > 1$, then for any isometric immersion of a doubly warped product $\phi : {}_{f_2}N_1 \times_{f_1} N_2$ into a Riemannian manifold M, the mean curvature H of ϕ satisfies*

$$H^2 > \frac{-4 n_1 n_2}{(n_1+n_2)^2} \max \tilde{K} \tag{26}$$

at some points.

In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then (26) *holds at some points.*

The proof of this theorem is similar to the proof of Theorem 15. However, because doubly warped products are somewhat different from ordinary warped products, we provide the proof of Theorem 5.1 as follows.

Proof of Theorem 27. Suppose contrary to (26), i.e., there were an isometric immersion ϕ whose mean curvature H satisfying

$$H^2 \leq \frac{-4n_1n_2}{(n_1+n_2)^2} \max \tilde{K}$$

everywhere on N, which gives

$$0 \leq -\frac{(n_1+n_2)^2}{4} H^2 - n_1n_2 \max \tilde{K}. \tag{27}$$

On the other hand, Theorem G would imply

$$-\frac{(n_1+n_2)^2}{4} H^2 - n_1n_2 \max \tilde{K} \leq n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2}.$$

After combining this with (27), we find

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \geq -\frac{(n_1+n_2)^2}{4} H^2 - n_1n_2 \max \tilde{K} \geq 0. \tag{28}$$

After applying the Average Method in PDE stated in Proposition 16 1), (28) shows that some f_i, $i = 1, 2$, could be constant *or* f_i would be two-imbalanced for every $q_i > 1$. This contradicts the assumption that f_i is nonconstant *and* two-balanced for some $q_i > 1$.

The last assertion follows from Definition 9, the first assertion of Theorems 27 and the fact that every L^q function has *two-finite*, *two-mild*, *two-obtuse*, *two-moderate*, *two-small* growth for the same q (cf. [18, Proposition 2.3]). ∎

In particular, if the ambient space M is of constant sectional curvature $c \leq 0$, then Theorem 27 reduces to the following.

Theorem 28. *If f_1, f_2 are nonconstant and* two-balanced *for some $q_1, q_2 > 1$, then for any isometric immersion of a doubly warped product $\phi : {}_{f_2}N_1 \times_{f_1} N_2$ into a Riemannian m-manifold $R^m(c)$ of constant curvature $c \leq 0$, the mean curvature H of ϕ satisfies*

$$H^2 > \frac{-4n_1n_2}{(n_1+n_2)^2} c \tag{29}$$

at some points.

In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then (29) *holds at some points.*

Also, the following are easy consequences of Theorem 27.

Corollary 29. *If for each j ($j = 1, 2$), f_j is nonconstant and* two-balanced *for some q_j, then there does not exist an isometric minimal immersion of any doubly warped product $\phi : {}_{f_2}N_1 \times_{f_1} N_2$ into any negatively curved Riemannian manifold. In particular, if each f_j is nonconstant and in L^{q_j} for some $q_j > 1$, then there does not exist isometric minimal immersion of $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ into a Euclidean space.*

As another easy applications of Theorem 27, we have the following dichotomy.

Corollary 30. *Suppose the squared mean curvature of the isometric immersion of a doubly warped product $\phi :_{f_2} N_1 \times_{f_1} N_2$ into a Riemannian manifold satisfies*

$$H^2 \leq \frac{-4n_1 n_2}{(n_1 + n_2)^2} \max \tilde{K} \tag{30}$$

everywhere on N. Then there exists an integer $i, 1 \leq i \leq 2$, such that the warping function f_i is either a constant or for every $q_i > 1$, f_i has two-imbalanced *growth.*

Analogously, by applying the growth estimates and the average method in PDE as before, we have the following Liouville property and a characterization result.

Corollary 31. *Suppose the squared mean curvature of an isometric immersion ϕ of a doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ into a Riemannian manifold satisfies* (27) *on N. If for each j ($j = 1, 2$), f_j is* two-balanced *for some $q_j > 1$, then we have*:

1) *Every warping function $f_j, 1 \leq j \leq 2$, is constant.*

2) *The isometric immersion ϕ is a minimal immersion into a Euclidean space.*

Corollary 32. *Let each f_j , $1 \leq j \leq 2$ be* two-balanced *for some $q_j > 1$. Then we have*:

1) *Every doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ does not admit an isometric minimal immersion into any negatively curved Riemannian manifold.*

2) *If N_2 is compact, then ${}_{f_2}N_1 \times_{f_1} N_2$ does not admit an isometric minimal immersion into a Euclidean space.*

We state a special case of Corollary 33

Corollary 33. *If each f_j ($j = 1, 2$) is in L^{q_j} for some $q_j > 1$, then we have:*

1) *Every doubly warped product* ${}_{f_2}N_1 \times_{f_1} N_2$ *does not admit an isometric minimal immersion into any negatively curved Riemannian manifold.*

2) *If* N_2 *is compact, then* ${}_{f_2}N_1 \times_{f_1} N_2$ *does not admit an isometric minimal immersion into a Euclidean space.*

Since Corollaries 29-33 can be in the same way as the proofs of Corollary 4, Theorem 3, Corollary 5 1) & 2), Corollary 6 and Corollary 7, we omit their proofs.

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