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ON THE SOLITON SOLUTIONS OF A FAMILY OF TZITZEICA EQUATIONS

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Abstract. We analyze several types of soliton solutions to a family of Tzitzeica equations. To this end we use two methods for deriving the soliton solutions: the dressing method and Hirota method. The dressing method allows us to derive two types of soliton solutions. The first type corresponds to a set of six symmetrically situated discrete eigenvalues of the Lax operator L; to each soliton of the second type one relates a set of twelve discrete eigenvalues of L. We also outline how one can construct general N soliton solution containing N_1 solitons of first type and N_2 solitons of second type, $N = N_1 + N_2$. The possible singularities of the solitons and the effects of change of variables that relate the different members of Tzitzeica family equations are briefly discussed. All equations allow quasi-regular as well as singular soliton solutions.

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Contents

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1. Introduction

In the present paper we continue our investigations of the famous equation due to the Romanian mathematician Gheorghe Tzitzeica[1], which we call now as Tzitzeica 1 equation [27, 28] and a closely related equation which we call Tzitzeica 2. In what follows we will denote them by T1 and T2. It was initially proposed as an equation describing special surfaces in differential geometry for which the ratio K/d^4 is constant, where K is the Gauss curvature of the surface and d is the distance from the origin to the tangent plane at the given point. Later on it turned out that the equation has wider importance, being nowadays used as an important evolutionary equation in nonlinear dynamics. The explicit form of T1 and T2 equations is

$$2\frac{\partial^2 \phi_1}{\partial \xi \partial \eta} = \mathrm{e}^{2\phi_1} - \mathrm{e}^{-4\phi_1}, \qquad 2\frac{\partial^2 \phi_2}{\partial \xi \partial \eta} = -(\mathrm{e}^{2\phi_2} - \mathrm{e}^{-4\phi_2}) \tag{1}$$

i.e., T1 and T2 have different signs at the right hand sides. The transition between T1 and T2 can be performed by several simple changes of variables (see below), some of which substantially modify the singularity properties of their solutions.

Tzitzeica equations attracted a lot of attention at the end of the '70s when for some time it was believed, that it is the only known equation, allowing a finite number of higher integrals of motion [8]. Soon however, it was proved that in fact, it possesses, like the other soliton equations, an infinite number of integrals of motion [30]. Next it was discovered that the equation has a hidden $\mathbb{Z}_3$ symmetry, which becomes evident in its Lax representation [21, 22]. This important discovery led Mikhailov to the notion of the reduction group and to the family of two-dimensional Toda field theories (TFT) related to the $\mathfrak{sl}(n)$ algebras [21]. Soon after it was established that: i) two-dimensional TFT can be related to any of the simple Lie algebras [9, 19, 23, 24], ii) other classes of integrable NLEE may also possess such symmetries [7, 9, 12, 13], and iii) the expansions over the squared solutions and the theory of their recursion operators can be constructed [15, 18, 29].

In previous papers [4, 5] we presented in the derivation of the soliton solutions of T1. Both versions of Tzitzeica equation allow Lax representation proposed by

[1] Actually the name of the famous Romanian mathematician contains the Romanian letter Ţ, which may be spelled as Tz. The factor 2 in equation (1) can be easily removed, but is kept for historical reasons.

Mikhailov [21, 22]. This allows one to apply the dressing method of Zakharov-Shabat-Mikhailov [21,32,33] for calculating their soliton solutions. In fact all these equations are particular examples of two-dimensional Toda field theories (TFT) [9,19,21,23,24]. They all can be solved exactly using the inverse scattering method [10, 16, 31].

In the present paper we start with the analysis of a more general class of equations, which we call Tzitzeica family equations. Their general form is

$$2\frac{\partial^2\phi}{\partial\xi\partial\eta} = \epsilon_1 c_1^2 \mathrm{e}^{2\phi} + \epsilon_2 c_2^2 \mathrm{e}^{-4\phi} \qquad (2)$$

where $\epsilon_1^2 = \epsilon_2^2 = 1$ and c_1 and c_2 are some positive real constants. Obviously equation T1 (respectively equation T2) is obtained from (2) by putting $\epsilon_1 = 1$, $\epsilon_2 = -1$, $c_1 = c_4 = 1$ (respectively $\epsilon_1 = -1$, $\epsilon_2 = 1$, $c_1 = c_4 = 1$). We will call T3 and T4 the equations

$$2\frac{\partial^2\phi_3}{\partial\xi\partial\eta} = -\mathrm{e}^{2\phi_3} - \mathrm{e}^{-4\phi_3}, \qquad 2\frac{\partial^2\phi_4}{\partial\xi\partial\eta} = \mathrm{e}^{2\phi_4} + \mathrm{e}^{-4\phi_4} \qquad (3)$$

which follow from (2) with $\epsilon_1 = \epsilon_2 = -1$, $c_1 = c_4 = 1$ and $\epsilon_1 = \epsilon_2 = 1$, $c_1 = c_4 = 1$ respectively.

The paper is organized as follows. In Section 2 we study a class of changes of variables that interrelate different members of Tzitzeica family. We shall see that equations T1 – T4 allow Lax representations so they can be solved exactly by the inverse scattering method, [6, 22]. In Section 3 the Zakharov-Shabat dressing method [33], adapted to systems with deep reductions [21, 22] is used to construct their soliton solutions. As a result we derive the soliton solutions of first and second types and analyze their singularities. Indeed, we find that even the simplest one-soliton solutions of first type may have an infinite number of singularities for finite values of ξ, η. Such singularities are characteristic also for other soliton-type equations, e.g. for Liouville equation [1, 2, 25, 26], for sinh-Gordon equation and others, see e.g. [11, 20, 25] and the references therein. At the same time, using an appropriate change of variables we obtain a solution having singularities at only two points which we call 'quasi-regular'. In Section 4 we outline how the dressing formalism can be extended to derive the N-soliton solution of the considered model with N_1 solitons of first type and N_2 solitons of second type, $N = N_1 + N_2$. In Section 5 we demonstrate how Hirota method can be applied for deriving the soliton solutions of Tzitzeica equations and show that it results compatible with the ones of the dressing method. In Section 6 we briefly outline the spectral properties of the Lax operators L. We demonstrate that the resolvent of L has pole singularities that coincide with the poles of the dressing factor and its inverse. We end by a discussion and conclusions.

2. Lorentz (Anti-)Invariance in Two-Dimensions

Obviously each of the TFT mentioned above can be viewed as a member of a hierarchy of NLEE which can be solved by applying the ISM to the corresponding Lax operator. However the Lorentz invariance singles out the TFT models from all the other members of NLEE in the hierarchy. Indeed, the TFT models allow changes of variables which may drastically change, as we shall demonstrate below, the properties of the soliton solutions.

2.1. Changes of Variables and the Lorentz (Anti-)Invariance

Let us now consider how simple linear change of variables

$$\vec{Y}' = A\vec{Y}, \qquad \vec{Y}' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}, \qquad \vec{Y} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4}$$

affect the solutions of Tzitzeica equations Obviously this transformations have to preserve, up to a sign, $\frac{\partial^2}{\partial\xi\partial\eta}$ which means that

$$A^T\sigma_1 A = \pm\sigma_1, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

which is equivalent to the relations

$$ac = bd = 0, \qquad ad + bc = \pm 1. \tag{6}$$

These relations are satisfied in two cases

$$1) \quad A_1^{\pm} = \begin{pmatrix} a & 0 \\ 0 & \pm 1/a \end{pmatrix}, \qquad 2) \quad A_2^{\pm} = \begin{pmatrix} 0 & b \\ \pm 1/b & 0 \end{pmatrix}. \tag{7}$$

Here a and b can be, in general, arbitrary complex numbers. However, below we will consider two cases: i) a and b – real and ii) a and b – purely imaginary.

Second class of transformations involves shifts of the field ϕ

$$\phi(\xi,\eta) \to \phi'(\xi,\eta) = \phi(\xi,\eta) - \ln c_0 + s_0\frac{\pi\mathrm{i}}{2} \tag{8}$$

where $c_0 > 0$ is a real constant and s_0 takes the values 0 and 1. If $s_0 = 0$ and $c_0 = c_1$ then T1 goes into

$$2\frac{\partial^2\phi_1'}{\partial\xi\partial\eta} = c_1^2\mathrm{e}^{2\phi_1'} - c_1^{-4}\mathrm{e}^{-4\phi_1'} \tag{9}$$

Table 1. Changes of variables that relate different members of Tzitzeica family equations.

	T1	T2	T3	T4
$A^{+}_{1,2}$, $s_0 = 0$	T1	T2	T3	T4
$A^{-}_{1,2}$, $s_0 = 0$	T2	T1	T4	T3
$A^{+}_{1,2}$, $s_0 = 1$	T3	T4	T1	T2
$A^{-}_{1,2}$, $s_0 = 1$	T4	T3	T2	T1

and similar expression for the T2 equation for ϕ_2, but with opposite signs for the terms in the right hand side.

If we now choose $s_0 = 1$ and $c_0 = c_1$ then T1 goes into

$$2\frac{\partial^2 \phi_1'}{\partial\xi\partial\eta} = -c_1^2 \mathrm{e}^{2\phi_1'} - c_1^{-4}\mathrm{e}^{-4\phi_1'} \tag{10}$$

which for $c_1 = 1$ coincides with T3 equation. We have listed the results of several such transformations in Table 1.

2.2. The Lax Representation of T2 Equation

Since different members of Tzitzeica family are related by changes of variables (see Table 1), then it will be enough to consider the Lax representation and soliton solutions of only one of them, say the second equation in (1) T2. It admits the following Lax representation

$$\begin{aligned} L_1\Psi(\xi,\eta,\lambda) &\equiv \mathrm{i}\,\frac{\partial\Psi(\xi,\eta,\lambda)}{\partial\xi} + 2\mathrm{i}\,\phi_\xi H_0\Psi(\xi,\eta,\lambda) + \lambda\mathcal{J}\Psi(\xi,\eta,\lambda) = 0 \\ L_2\Psi(\xi,\eta,\lambda) &\equiv \mathrm{i}\,\frac{\partial\Psi(\xi,\eta,\lambda)}{\partial\eta} + \lambda^{-1}V_1\Psi(\xi,\eta,\lambda) = 0 \end{aligned} \tag{11}$$

where

$$H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V_1(\xi,\eta) = \begin{pmatrix} 0 & 0 & \mathrm{e}^{-4\phi} \\ \mathrm{e}^{2\phi} & 0 & 0 \\ 0 & \mathrm{e}^{2\phi} & 0 \end{pmatrix}. \tag{12}$$

The reductions of the Lax pair for T2 equation are similar but not the same as for the well known T1 equation [4]

1. $\mathbb{Z}_3$-reduction

$$Q^{-1}\Psi(\xi,\eta,\lambda)Q = \Psi(\xi,\eta,q\lambda), \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad q = \mathrm{e}^{2\pi\mathrm{i}/3} \tag{13}$$

which restricts H_0, $\mathcal{J}$ and V_1 by

$$Q^{-1}H_0Q = H_0, \qquad Q^{-1}\mathcal{J}Q = q\mathcal{J}, \qquad Q^{-1}V_1Q = q^{-1}V_1. \tag{14}$$

These conditions are satisfied identically.

2. First $\mathbb{Z}_2$-reduction

$$\Psi^*(\xi,\eta,-\lambda^*) = \Psi(\xi,\eta,\lambda) \tag{15}$$

i.e.,

$$H_0 = H_0^*, \qquad J_1 = J_1^*, \qquad V_1 = V_1^* \tag{16}$$

which means that $\phi = \phi^*$.

3. Second $\mathbb{Z}_2$-reduction

$$A_0^{-1}\Psi^\dagger(\xi,\eta,\lambda^*)A_0 = \Psi^{-1}(\xi,\eta,\lambda), \qquad A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{17}$$

i.e.,

$$A_0^{-1}H_0^\dagger A_0 = -H_0, \qquad A_0^{-1}\mathcal{J}^\dagger A_0 = \mathcal{J}, \qquad A_0^{-1}V_1^\dagger A_0 = V_1. \tag{18}$$

3. The Dressing Method and Dressing Factors for T2 Equation

Let us start with a Lax representation of the form

$$L_{10}\Psi_0 \equiv \mathrm{i}\,\frac{\partial\Psi_0}{\partial\xi} + \lambda\mathcal{J}\Psi_0 = 0, \qquad L_{20}\Psi_0 \equiv \mathrm{i}\,\frac{\partial\Psi_0}{\partial\eta} + \lambda^{-1}\mathcal{J}^2\Psi_0 = 0. \tag{19}$$

The fundamental solution $\Psi_0(\xi,\eta,\lambda)$, known also as the 'naked' solution, has as potential the trivial solution of T2 equation: $\phi_0(\xi,\eta) = 0$

The 'dressed' Lax pair, given by (11), admits the "dressed" fundamental solution $\Psi(\xi,\eta,\lambda)$, with the potential the nontrivial solution $\phi(\xi,\eta)$.

The fundamental solutions Ψ and Ψ_0 are related by the dressing factor $u(\xi,\eta,\lambda)$

$$\Psi(\xi,\eta,\lambda) = u(\xi,\eta,\lambda)\Psi_0(\xi,\eta,\lambda)u_+^{-1}(\lambda), \qquad u_+(\lambda) = \lim_{\xi\to\infty} u(\xi,\eta,\lambda) \tag{20}$$

which means that $u(\xi,\eta,\lambda)$ must satisfy

$$\begin{aligned} \mathrm{i}\,\frac{\partial u}{\partial \xi} + 2\mathrm{i}\,\phi_\xi H_0 u(\xi,\eta,\lambda) + \lambda\,[\mathcal{J}, u(\xi,\eta,\lambda)] &= 0 \\ \mathrm{i}\,\frac{\partial u}{\partial \eta} + \frac{1}{\lambda} V_1 u(\xi,\eta,\lambda) - \frac{1}{\lambda} u(\xi,\eta,\lambda)\mathcal{J}^2 &= 0. \end{aligned} \tag{21}$$

Since both Lax pairs (the dressed (11) and the naked one (19)) satisfy the three reductions, then also the dressing factor must satisfy them

$$\begin{aligned} &\text{a)} \quad Q^{-1}u(\xi,\eta,\lambda)Q = u(\xi,\eta,q\lambda), \qquad \text{b)} \quad u^*(\xi,\eta,-\lambda^*) = u(\xi,\eta,\lambda) \\ &\text{c)} \quad A_0^{-1}u^\dagger(\xi,\eta,\lambda^*)A_0 = u^{-1}(\xi,\eta,\lambda) \end{aligned} \tag{22}$$

where A_0 is defined by equation (17).

3.1. One Soliton Solution of First Type

A natural anzatz for the dressing factor with simple poles in λ is [21]

$$u(\xi,\eta,\lambda) = \mathbb{1} + \frac{1}{3}\left(\frac{A_1}{\lambda-\lambda_1} + \frac{Q^{-1}A_1Q}{\lambda q^2-\lambda_1} + \frac{Q^{-2}A_1Q^2}{\lambda q-\lambda_1}\right) \tag{23}$$

where $A_1(\xi,\eta)$ is a 3×3 degenerate matrix of the form

$$A_1(\xi,\eta) = |n(\xi,\eta)\rangle\langle m^T(\xi,\eta)|, \qquad (A_1)_{ij}(\xi,\eta) = n_i(\xi,\eta)m_j(\xi,\eta). \tag{24}$$

The first reduction (22a) on $u(x,t,\lambda)$ is automatically satisfied by the anzatz (23). The second reduction (22b) leads to

$$\frac{\eta_{j-k}n_k m_j}{\lambda^3-\lambda_1^{*,3}} = -\frac{\rho_{j-k}n_k^* m_j^*}{\lambda^3+\lambda_1^{*,3}}. \tag{25}$$

Here and below $j-k$ is understood modulo 3 and

$$\begin{aligned} &\eta_0 = \lambda_1^2, \qquad &&\eta_1 = \lambda\lambda_1, \qquad &&\eta_2 = \lambda^2 \\ &\rho_0 = \lambda_1^{*,2}, \qquad &&\rho_1 = -\lambda\lambda_1^*, \qquad &&\rho_2 = \lambda^2. \end{aligned} \tag{26}$$

In addition we must have

$$\lambda_1^{*,3} = -\lambda_1^3$$

and

$$\lambda_1^{*,2}n_i^*m_i^* = -\lambda_1^2 n_i m_i, \quad \lambda_1^* n_i^* m_{i+1}^* = \lambda_1 n_i m_{i+1}, \quad n_i^* m_{i+2}^* = -n_i m_{i+2}$$

where again all matrix indices are understood modulo 3. These relations can be rewritten as

$$
\begin{aligned}
\arg n_i + \arg m_i &= \frac{\pi}{2} - 2\arg\lambda_1, \qquad \arg n_i + \arg m_{i+1} = -\arg\lambda_1 \\
\arg n_i + \arg m_{i+2} &= \frac{\pi}{2}, \qquad \arg\lambda_1 = (2k+1)\frac{\pi}{6}, \qquad k = 0, 1, ..., 5.
\end{aligned} \tag{27}
$$

So we can consider with no limitations that $\lambda_1 = -\lambda_1^*$ and $A_1 = -A_1^*$. More specifically we will assume that the vector $\langle m^T(\xi,\eta)|$ is real, while the vector $|n(\xi,\eta)\rangle$ has purely imaginary components.

The third reduction (22c) on $u(x,t,\lambda)$ can be put in the form

$$
u(\xi,\eta,\lambda)A_0^{-1}u^\dagger(\xi,\eta,\lambda^*)A_0 = \mathbb{1}. \tag{28}
$$

Let us now multiply (28) by $\lambda - \lambda_1$, take the limit $\lambda \to \lambda_1$ and take into account equation (14). This gives

$$
m_k = \frac{n_{4-k}^*}{\lambda_1^3 - \lambda_1^{*,3}} \sum_{k=1}^{3} \kappa_{s-k} m_s m_{4-s}^* \tag{29}
$$

where

$$
\kappa_0 = \lambda_1^{*,2}, \qquad \kappa_1 = \lambda_1^2, \qquad \kappa_2 = \lambda_1\lambda_1^*. \tag{30}
$$

Thus, taking into account that $\lambda_1 = \mathrm{i}\,\rho_1$, ρ_1 – real and $m_k = m_k^*$, we obtain

$$
\begin{aligned}
n_1 &= \frac{2\lambda_1^3 m_3^*}{\lambda_1^2 m_3^* m_1 + |\lambda_1|^2|m_2|^2 + \lambda_1^{2,*} m_1^* m_3} = \frac{2\mathrm{i}\,\rho_1 m_3}{2m_1m_3 - m_2^2} \\
n_2 &= \frac{2\lambda_1^3 m_2^*}{\lambda_1^{2,*} m_3^* m_1 + \lambda_1^2|m_2|^2 + |\lambda_1|^2 m_1^* m_3} = \frac{2\mathrm{i}\,\rho_1}{m_2} \\
n_3 &= \frac{2\lambda_1^3 m_1^*}{|\lambda_1|^2 m_3^* m_1 + \lambda_1^{2,*}|m_2|^2 + \lambda_1^2 m_1^* m_3} = \frac{2\mathrm{i}\,\rho_1 m_1}{m_2^2}.
\end{aligned} \tag{31}
$$

In order to obtain the vectors $|n\rangle$ and $\langle m^T|$ in terms of ξ and η we first impose the limit $\lambda \to \lambda_1$ in equation (21). We obtain that the residue A_1 must satisfy

$$
\begin{aligned}
\mathrm{i}\,\frac{\partial A_1}{\partial \xi} + 2\mathrm{i}\,\phi_\xi H_0 A_1 + \lambda_1[\mathcal{J}, A_1] = 0 \\
\mathrm{i}\,\frac{\partial A_1}{\partial \eta} + \lambda_1^{-1} V_1 A_1 - \lambda_1^{-1} A_1 \mathcal{J}^2 = 0.
\end{aligned} \tag{32}
$$

Since $A_1 = |n\rangle\langle m^T|$ we find that (32) is satisfied if

$$\begin{aligned} &\mathrm{i}\,\frac{\partial|n\rangle}{\partial\xi} + 2\mathrm{i}\,\phi_\xi H_0|n\rangle + \lambda_1\mathcal{J}|n\rangle = 0, \qquad \mathrm{i}\,\frac{\partial\langle m^T|}{\partial\xi} - \lambda_1\langle m^T|\mathcal{J} = 0 \\ &\mathrm{i}\,\frac{\partial|n\rangle}{\partial\eta} + \lambda_1^{-1}V_1|n\rangle = 0, \qquad \mathrm{i}\,\frac{\partial\langle m^T|}{\partial\eta} - \lambda_1^{-1}\langle m^T|\mathcal{J}^2 = 0 \end{aligned} \tag{33}$$

i.e.,

$$|n\rangle = \Psi(\xi,\eta,\mathrm{i}\,\rho_1)|n_0\rangle, \qquad \langle m^T| = \langle m_0^T|(\Psi_0)^{-1}(\xi,\eta,\mathrm{i}\,\rho_1) \tag{34}$$

which means that $|n(\xi,\eta)\rangle$ is an eigenfunction for the "dressed" Lax pair L_1, L_2, while $\langle m^T(\xi,\eta)|$ is an eigenfunction for the "naked" Lax pair L_{10}, L_{20}.

From (19), using direct calculation we obtain

$$\begin{gathered} \Psi_0(\xi,\eta,\lambda) = f_0\mathrm{e}^{\mathrm{i}\,\lambda J\xi + \mathrm{i}\,\lambda^{-1}J^2\eta}f_0^{-1} \\ f_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} q & 1 & q^2 \\ 1 & 1 & 1 \\ q^2 & 1 & q \end{pmatrix}, \quad f_0^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} q^2 & 1 & q \\ 1 & 1 & 1 \\ q & 1 & q^2 \end{pmatrix}, \quad J = \mathrm{diag}\,(q^2, 1, q) \end{gathered} \tag{35}$$

which means that

$$\langle m^T| = \langle m_0^T|f_0\mathrm{e}^{\rho_1 J\xi - \rho_1^{-1}J^2\eta}f_0^{-1}. \tag{36}$$

Using the notation

$$\langle m_0^T|\frac{1}{\sqrt{3}}f_0 = (\mu_{01}, \mu_{02}, \mu_{03}) \tag{37}$$

we obtain the following explicit forms for the components of vector $\langle m^T(\xi,\eta)|$

$$\begin{aligned} m_1(\xi,\eta) &= q^2\mu_{01}\mathrm{e}^{-\mathcal{X}_1}\mathrm{e}^{-\mathrm{i}\,\Omega_1} + \mu_{02}\mathrm{e}^{2\mathcal{X}_1} + q\mu_{03}\mathrm{e}^{-\mathcal{X}_1}\mathrm{e}^{\mathrm{i}\,\Omega_1} \\ m_2(\xi,\eta) &= \mu_{01}\mathrm{e}^{-\mathcal{X}_1}\mathrm{e}^{-\mathrm{i}\,\Omega_1} + \mu_{02}\mathrm{e}^{2\mathcal{X}_1} + \mu_{03}\mathrm{e}^{-\mathcal{X}_1}\mathrm{e}^{\mathrm{i}\,\Omega_1} \\ m_3(\xi,\eta) &= q\mu_{01}\mathrm{e}^{-\mathcal{X}_1}\mathrm{e}^{-\mathrm{i}\,\Omega_1} + \mu_{02}\mathrm{e}^{2\mathcal{X}_1} + q^2\mu_{03}\mathrm{e}^{-\mathcal{X}_1}\mathrm{e}^{\mathrm{i}\,\Omega_1} \end{aligned} \tag{38}$$

where

$$\mathcal{X}_1 = \frac{1}{2}\left(\rho_1\xi - \frac{\eta}{\rho_1}\right), \qquad \Omega_1 = \frac{\sqrt{3}}{2}\left(\rho_1\xi + \frac{\eta}{\rho_1}\right). \tag{39}$$

For $\mu_{0,1} = \mu_{0,3}^* = |\mu_{01}|\mathrm{e}^{\mathrm{i}\,\alpha_0}$ and $\mu_{0,2} = \mu_{0,2}^*$ we can rewrite m_i from (38) as the following real-valued functions

$$\begin{aligned} m_1(\xi,\eta) &= \mu_{02}\mathrm{e}^{2\mathcal{X}_1} + 2|\mu_{01}|\mathrm{e}^{-\mathcal{X}_1}\cos\left(\Omega_1 - \alpha_{01} + \frac{2\pi}{3}\right) \\ m_2(\xi,\eta) &= \mu_{02}\mathrm{e}^{2\mathcal{X}_1} + 2|\mu_{01}|\mathrm{e}^{-\mathcal{X}_1}\cos\left(\Omega_1 - \alpha_{01}\right) \\ m_3(\xi,\eta) &= \mu_{02}\mathrm{e}^{2\mathcal{X}_1} + 2|\mu_{01}|\mathrm{e}^{-\mathcal{X}_1}\cos\left(\Omega_1 - \alpha_{01} - \frac{2\pi}{3}\right). \end{aligned} \tag{40}$$

The components of the vector $|n\rangle$ in (31) become

$$n_1 = \frac{2\mathrm{i}\,\rho_1 m_3}{2m_1 m_3 - m_2^2}, \qquad n_2 = \frac{2\mathrm{i}\,\rho_1}{m_2}, \qquad n_3 = \frac{2\mathrm{i}\,\rho_1 m_1}{m_2^2}. \tag{41}$$

In order to obtain the solution of T2 equation we impose the limit $\lambda \to 0$ in (21) with the result

$$2\phi_\xi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\frac{\partial u}{\partial \xi} u^{-1}(\xi, \eta, 0) \tag{42}$$

where

$$u(\xi,\eta,0) = \mathbb{1} - \frac{1}{3\lambda_1}(A_1 + Q^{-1}A_1Q + Q^{-2}A_1Q^2) = \left(1 - \frac{1}{\lambda_1}A_{1,jk}\right)\delta_{jk} \tag{43}$$

which means that

$$2\phi_\xi = -\frac{\partial u_{0;11}}{\partial \xi}\frac{1}{u_{0;11}} = -\frac{\partial}{\partial \xi}\ln u_{0;11} \tag{44}$$

or

$$2\phi(\xi,\eta) = -\ln\left|1 - \frac{n_1 m_1}{\lambda_1}\right| = \ln\left|\frac{2m_1m_3 - m_2^2}{m_2^2}\right|. \tag{45}$$

After introducing m_i from (40) we obtain the one-soliton solution of the first type for $\lambda_1 = \mathrm{i}\,\rho_1$

$$\phi_{1s}(\xi,\eta) = \frac{1}{2}\ln\left|\frac{|\mu_{01}|^2 \mathrm{e}^{-3\mathcal{X}_1}\left(4\cos^2(\tilde{\Omega}_1) - 6\right) - 8|\mu_{01}|\mu_{02}\cos(\tilde{\Omega}_1) + \mu_{02}^2\mathrm{e}^{3\mathcal{X}_1}}{4|\mu_{01}|^2\mathrm{e}^{-3\mathcal{X}_1}\cos^2(\tilde{\Omega}_1) + 4|\mu_{01}|\mu_{02}\cos(\tilde{\Omega}_1) + \mu_{02}^2\mathrm{e}^{3\mathcal{X}_1}}\right| \tag{46}$$

where $\tilde{\Omega}_1 = \Omega_1 - \alpha_{01}$. We observe that this is not a travelling wave solution. Only if we take the limit $\mu_{02} \to 0$ we obtain a travelling wave solution of the form

$$\phi(\xi,\eta) = \mathrm{i}\,\frac{\pi}{2} + \frac{1}{2}\ln\left[\frac{3}{2}\tan^2\left(\frac{\sqrt{3}}{2}(\rho_1\xi + \rho_1^{-1}\eta) - \alpha_{01}\right) + \frac{1}{2}\right]. \tag{47}$$

The solution is singular and it blows up for $\frac{\sqrt{3}}{2}(\rho_1\xi + \rho_1^{-1}\eta) - \alpha_{01} = (2k+1)\pi/2$, $k = 0, \pm 1, \ldots$.

For $\alpha_{01} \to \alpha_{01} + \frac{\pi}{2}$ ($m_{10}, m_{20}, m_{30} \in \mathbb{C}$ and they are purely imaginary), the solution (47) becomes

$$\phi(\xi,\eta) = \mathrm{i}\,\frac{\pi}{2} + \frac{1}{2}\ln\left[\frac{3}{2}\cot^2\left(\frac{\sqrt{3}}{2}(\rho_1\xi + \rho_1^{-1}\eta) - \alpha_{01}\right) + \frac{1}{2}\right]. \tag{48}$$

The above solution is also singular and it blows up for $\frac{\sqrt{3}}{2}(\rho_1\xi + \rho_1^{-1}\eta) + \alpha_0 = k\pi$, $k = 0, \pm 1, \ldots$.

Remark 1. *It is easy to check, that the real parts of* $\phi(\xi,\eta)$ *in equations* (47) *and* (48) *are in fact solutions to* T4 *equation.*

In order to get 'quasi-regular' solutions of T2 equation, we can apply the changes of variables A_1^+ with $a = \mathrm{i}$ or A_2^+ with $b = \mathrm{i}$. This gives the following solutions expressed in terms of hyperbolic functions

$$\phi(\xi,\eta) = \frac{1}{2}\ln\left[\frac{3}{2}\tanh^2\left(\frac{\sqrt{3}}{2}(\rho_1\xi - \rho_1^{-1}\eta) - \alpha_{01}\right) - \frac{1}{2}\right] \tag{49}$$

and

$$\phi(\xi,\eta) = \frac{1}{2}\ln\left[\frac{3}{2}\coth^2\left(\frac{\sqrt{3}}{2}(\rho_1\xi - \rho_1^{-1}\eta) - \alpha_{01}\right) - \frac{1}{2}\right] \tag{50}$$

which are singular at the points for which

$$\tanh\left(\frac{\sqrt{3}}{2}(\rho_1\xi - \rho_1^{-1}\eta) - \alpha_{01}\right) = \pm\frac{1}{\sqrt{3}}$$

or

$$\coth\left(\frac{\sqrt{3}}{2}(\rho_1\xi - \rho_1^{-1}\eta) - \alpha_{01}\right) = \pm\frac{1}{\sqrt{3}}$$

respectively. These solutions have also been found by Mikhailov in [21]. As compared with the previous solutions, that have an infinite number of singularities, these ones have singularities at only two points. That is why we took the liberty to call them 'quasi-regular'.

3.2. The Singularity Properties of the Soliton Solutions

Here we will discuss the types of singularities of the one-soliton solutions and how they are influenced by the changes of variables. As we already mentioned, the singularities in the soliton solutions are not rare, see [11, 20].

Let us first see how the changes of variables affect the Lax representation (11) and, as a consequence, how they affect the fundamental solution. We will be particularly interested in the properties of the 'naked' Lax pair and its fundamental solution $\Psi_0(\xi,\eta,\lambda)$. This comes from the fact, that the soliton solution is constructed as a rational function of the elements of $\Psi_0(\xi,\eta,\lambda)$.

Let us start with the change of variables A_1^+. Here the situation is simple as we readily get

$$\begin{aligned} L_1(\lambda) &\to \frac{1}{a}L_1(\lambda/a), \qquad L_2(\lambda)\to aL_2(a\lambda) \\ \Psi_0(\xi',\eta',\lambda') &\to \Psi_0\left(a\xi,\frac{\eta}{a},\frac{\lambda}{a}\right). \end{aligned} \tag{51}$$

In other words this change of variables leaves invariant the compatibility of the Lax pair, so obviously it will map a solution of T2 into a solution of T2. However now we have to keep in mind, that the change of variables must be extended also to the spectral parameter $\lambda\to\lambda/a$ and, of course to the discrete eigenvalues of $L_{1,2}$: $\lambda_1\to\lambda_1/a$ and therefore $\rho_1\to\rho_1/a$.

In particular, from equation (39) we see, that

$$\begin{aligned} X_1'(\xi',\eta',\lambda_1') &= \frac{1}{2}\left(\lambda_1'\xi'+\frac{\eta'}{\lambda_1'}\right) = X_1(\xi,\eta,\lambda_1) \\ \Omega_1'(\xi',\eta',\lambda_1') &= \frac{1}{2}\left(\lambda_1'\xi'+\frac{\eta'}{\lambda_1'}\right) = \Omega_1(\xi,\eta,\lambda_1) \end{aligned} \tag{52}$$

i.e., X_1 and Ω_1 are invariant with respect to A_1^+ transformations provided

$$\lambda_1' = \frac{\lambda_1}{a}\cdot \tag{53}$$

Now it is a bit more interesting to analyze the changes A_2^+

$$\begin{aligned} L_1''(\lambda) &\equiv b\left(\mathrm{i}\,\frac{\partial}{\partial\eta''}+2\mathrm{i}\,\phi_{\eta''}H_0+\lambda\mathcal{J}\right)\Psi(\xi'',\eta'',\lambda)=0 \\ L_2''(\lambda) &\equiv b\left(\mathrm{i}\,\frac{\partial}{\partial\xi''}+\frac{1}{\lambda b}V_1(\xi'',\eta'')\right)\Psi(\xi'',\eta'',\lambda)=0. \end{aligned} \tag{54}$$

Let us apply a gauge transformation, i.e., change from $\Psi(\xi'',\eta'',\lambda)$ to

$$\tilde{\Psi}(\xi'',\eta'',\lambda)A_0\mathrm{e}^{2\phi H_0}\Psi(\xi'',\eta'',\lambda'') \tag{55}$$

where H_0 and A_0 are defined in equations (16) and (17) respectively. This gives us

$$L_1''(\lambda'') = L_2(\lambda), \qquad L_2''(\lambda'') = L_1(\lambda), \qquad \lambda'' = \frac{1}{b\lambda}\cdot \tag{56}$$

So the A_2^+ change is equivalent to interchanging the Lax operators L_1 and L_2, which again preserves their compatibility condition. Applied to X_1 and Φ_1 these transformations lead to

$$\Psi_0''(\xi', \eta', \lambda_1'') = A_0 \Psi_0(\xi, \eta, \lambda_1) A_0. \tag{57}$$

Of course, analyzing the fundamental solutions we have to pay attention also whether the parameters a and b are real or purely imaginary. In addition we have to take into account, that λ_1 could be purely imaginary as above, but for other cases it could also be real. It is precisely this choice of the parameters a, b and λ_1 that may change the singularity properties of the solutions.

3.3. One Soliton Solutions of Second Type

Our anzatz for the dressing factor is

$$\begin{aligned} u(\xi, \eta, \lambda) = \mathbb{1} &+ \frac{1}{3}\left(\frac{A_1}{\lambda - \lambda_1} + \frac{Q^{-1}A_1 Q}{\lambda q^2 - \lambda_1} + \frac{Q^{-2}A_1Q^2}{\lambda q - \lambda_1}\right) \\ &- \frac{1}{3}\left(\frac{A_1^*}{\lambda + \lambda_1^*} + \frac{Q^{-1}A_1^* Q}{\lambda q^2 + \lambda_1^*} + \frac{Q^{-2}A_1^*Q^2}{\lambda q + \lambda_1^*}\right) \end{aligned} \tag{58}$$

which obviously satisfies the $\mathbb{Z}_3$-reduction and the first $\mathbb{Z}_2$-reduction.

In order to find how the components of the vector $|n\rangle$ are expressed in terms of the vector $|m^T\rangle$ we use the same procedure as in the three-poles case. First we rewrite the dressing factor in the following form

$$u(\xi, \eta, \lambda) = \mathbb{1} + \frac{1}{\lambda^3 - \lambda_1^3}\mathcal{A}_1(\xi, \eta, \lambda) - \frac{1}{\lambda^3 + \lambda_1^{3,*}}\mathcal{A}_1^*(\xi, \eta, -\lambda^*) \tag{59}$$

where

$$\begin{aligned} \mathcal{A}_1(\xi, \eta, \lambda) &= \begin{pmatrix} \eta_0 n_1 m_1 & \eta_1 n_1 m_2 & \eta_2 n_1 m_3 \\ \eta_2 n_2 m_1 & \eta_0 n_2 m_2 & \eta_1 n_2 m_3 \\ \eta_1 n_3 m_1 & \eta_2 n_3 m_2 & \eta_0 n_3 m_3 \end{pmatrix} \\ \mathcal{A}_1^*(\xi, \eta, -\lambda^*) &= \begin{pmatrix} \rho_0 n_1^* m_1^* & \rho_1 n_1^* m_2^* & \rho_2 n_1^* m_3^* \\ \rho_2 n_2^* m_1^* & \rho_0 n_2^* m_2^* & \rho_1 n_2^* m_3^* \\ \rho_1 n_3^* m_1^* & \rho_2 n_3^* m_2^* & \rho_0 n_3^* m_3^* \end{pmatrix} \end{aligned} \tag{60}$$

with

$$\begin{aligned} \eta_0 &= \lambda_1^2, & \eta_1 &= \lambda\lambda_1, & \eta_2 &= \lambda^2 \\ \rho_0 &= \lambda_1^{*,2}, & \rho_1 &= -\lambda\lambda_1^*, & \rho_2 &= \lambda^2. \end{aligned} \tag{61}$$

We insert the dressing factor $u(\xi,\eta,\lambda)$ into the second $\mathbb{Z}_2$-reduction, multiply by $\lambda-\lambda_1$, and take the limit $\lambda\to\lambda_1$ in order to obtain

$$\langle m^T|A_0 = \langle m^T|A_0\left[-\frac{1}{\lambda_1^3-\lambda_1^{*,3}}\mathcal{A}_1^\dagger(\lambda_1^*)+\frac{1}{2\lambda_1^3}\mathcal{A}_1^T(-\lambda_1)\right]. \qquad (62)$$

After direct calculation we obtain

$$m_3=\zeta_1 K_1 n_1^* + c_1 P_1 n_1, \quad m_2=\zeta_1 K_2 n_2^* + c_1 P_2 n_2, \quad m_1=\zeta_1 K_3 n_3^* + c_1 P_3 n_3$$

where

$$\begin{aligned}
&K_1=\lambda_1^{*,2}m_3m_1^*+\lambda_1\lambda_1^*m_2m_2^*+\lambda_1^2m_1m_3^*, && P_1=2m_1m_3-m_2^2\\
&K_2=\lambda_1^{2}m_3m_1^*+\lambda_1^{*,2}m_2m_2^*+\lambda_1\lambda_1^*m_1m_3^*, && P_2=m_2^2\\
&K_3=\lambda_1\lambda_1^*m_3m_1^*+\lambda_1^2m_2m_2^*+\lambda_1^{*,2}m_1m_3^*, && P_3=m_2^2\\
&\zeta_1=-\frac{1}{\lambda_1^3-\lambda_1^{*,3}}, && c_1=\frac{1}{2\lambda_1}.
\end{aligned} \qquad (63)$$

We rewrite the above result in a matrix form

$$|\mu\rangle=\begin{pmatrix} m_3\\ m_2\\ m_1\\ \hline m_3^*\\ m_2^*\\ m_1^*\end{pmatrix}, \qquad |\nu\rangle=\begin{pmatrix} n_1\\ n_2\\ n_3\\ \hline n_1^*\\ n_2^*\\ n_3^*\end{pmatrix}, \qquad |\mu\rangle=\mathcal{M}|\nu\rangle \qquad (64)$$

where

$$\mathcal{M}=\left(\begin{array}{ccc|ccc}
c_1P_1 & 0 & 0 & \zeta_1K_1 & 0 & 0\\
0 & c_1P_2 & 0 & 0 & \zeta_1K_2 & 0\\
0 & 0 & c_1P_3 & 0 & 0 & \zeta_1K_3\\ \hline
\zeta_1K_1^* & 0 & 0 & c_1P_1^* & 0 & 0\\
0 & \zeta_1K_2^* & 0 & 0 & c_1P_2^* & 0\\
0 & 0 & \zeta_1K_3^* & 0 & 0 & c_1P_3^*
\end{array}\right). \qquad (65)$$

The result is

$$|\nu\rangle=\mathcal{M}^{-1}|\nu\rangle$$

$$\mathcal{M}^{-1}=\left(\begin{array}{ccc|ccc}
-c_1^*\tilde{P}_1^* & 0 & 0 & \zeta_1\tilde{K}_1 & 0 & 0\\
0 & -c_1^*\tilde{P}_2^* & 0 & 0 & \zeta_1\tilde{K}_2 & 0\\
0 & 0 & -c_1^*\tilde{P}_3^* & 0 & 0 & \zeta_1\tilde{K}_3\\ \hline
\zeta_1^*\tilde{K}_1^* & 0 & 0 & -c_1\tilde{P}_1 & 0 & 0\\
0 & \zeta_1^*\tilde{K}_2^* & 0 & 0 & -c_1\tilde{P}_2 & 0\\
0 & 0 & \zeta_1^*\tilde{K}_3^* & 0 & 0 & -c_1\tilde{P}_3
\end{array}\right) \qquad (66)$$

where

$$\tilde{P}_s^* = \frac{P_s^*}{d_s}, \qquad \tilde{P}_s = \frac{P_s}{d_s}, \qquad \tilde{K}_s = \frac{K_s}{d_s}, \qquad \tilde{K}_s^* = \frac{K_s^*}{d_1}$$

$$\begin{aligned} d_1 &= \zeta_1\zeta_1^* K_1 K_1^* - c_1 c_1^* P_1 P_1^* \\ d_2 &= \zeta_1\zeta_1^* K_2 K_2^* - c_1 c_1^* P_2 P_2^* \\ d_3 &= \zeta_1\zeta_1^* K_3 K_3^* - c_1 c_1^* P_3 P_3^*. \end{aligned} \tag{67}$$

From the above equations we obtain $|n\rangle$ in terms of $\langle m^T|$

$$\begin{aligned} n_1 &= \frac{1}{d_1}(-c_1^* P_1^* m_3 + \zeta_1 K_1 m_3^*) \\ n_2 &= \frac{1}{d_2}(-c_1^* P_2^* m_2 + \zeta_1 K_2 m_2^*) \\ n_3 &= \frac{1}{d_3}(-c_1^* P_3^* m_1 + \zeta_1 K_3 m_1^*). \end{aligned} \tag{68}$$

In this case we choose a general form for the poles: $\lambda_1 = \rho_1 \mathrm{e}^{\mathrm{i}\,\beta_1}$. Without restrictions we can choose $0 < \beta_1 < \frac{\pi}{6}$ and determine the expressions of $\langle m^T|$ as

$$\begin{aligned} m_1 &= q^2\mu_{01}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_1 - \mathcal{Y}_1} + \mu_{02}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_2 - \mathcal{Y}_2} + q\mu_{03}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_3 - \mathcal{Y}_3} \\ m_2 &= \mu_{01}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_1 - \mathcal{Y}_1} + \mu_{02}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_2 - \mathcal{Y}_2} + \mu_{03}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_3 - \mathcal{Y}_3} \\ m_3 &= q\mu_{01}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_1 - \mathcal{Y}_1} + \mu_{02}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_2 - \mathcal{Y}_2} + q^2\mu_{03}\mathrm{e}^{\mathrm{i}\,\mathcal{X}_3 - \mathcal{Y}_3} \end{aligned} \tag{69}$$

where

$$\begin{aligned} \mathcal{X}_1 &= -\left(\xi\rho_1 + \frac{\eta}{\rho_1}\right)\cos\left(\beta_1 - \frac{2\pi}{3}\right), & \mathcal{Y}_1 &= -\left(\xi\rho_1 - \frac{\eta}{\rho_1}\right)\sin\left(\beta_1 - \frac{2\pi}{3}\right) \\ \mathcal{X}_2 &= -\left(\xi\rho_1 + \frac{\eta}{\rho_1}\right)\cos\left(\beta_1\right), & \mathcal{Y}_2 &= -\left(\xi\rho_1 - \frac{\eta}{\rho_1}\right)\sin\left(\beta_1\right) \\ \mathcal{X}_3 &= -\left(\xi\rho_1 + \frac{\eta}{\rho_1}\right)\cos\left(\beta_1 + \frac{2\pi}{3}\right), & \mathcal{Y}_3 &= -\left(\xi\rho_1 - \frac{\eta}{\rho_1}\right)\sin\left(\beta_1 + \frac{2\pi}{3}\right). \end{aligned} \tag{70}$$

We determine the one-soliton solution for the second kind of solitons using exactly the same technique

$$\Phi = -\frac{1}{2}\ln\left|1 - \frac{1}{\lambda_1}n_1 m_1 - \frac{1}{\lambda_1^*}n_1^* m_1^*\right|. \tag{71}$$

4. The Generic N-Soliton Solution for T2 Equation

Let us consider the dressing factor of the following form

$$u(\xi,\eta,\lambda) = \mathbb{1} + \frac{1}{3}\sum_{s=0}^{2}\left(\sum_{l=1}^{N_1}\frac{Q^{-s}A_l Q^s}{\lambda q^s - \lambda_l} + \sum_{r=N_1+1}^{N}\left(\frac{Q^{-s}A_r Q^s}{q^s\lambda - \lambda_r} - \frac{Q^{-s}A_r^* Q^s}{\lambda q^s + \lambda_r^*}\right)\right) \tag{72}$$

with $3N_1 + 6N_2$ complex poles from which N_1 are purely imaginary, satisfying the condition $\lambda_p = -\lambda_p^*$.

Then we write down the residues $A_k(\xi,\eta)$ as degenerate 3×3 matrices of the form

$$A_k(\xi,\eta) = |n_k(\xi,\eta)\rangle\langle m_k^T(\xi,\eta)|, \qquad (A_k)_{ij}(\xi,\eta) = n_{ki}(\xi,\eta)m_{kj}(\xi,\eta). \tag{73}$$

From the second $\mathbb{Z}_2$-reduction, $A_0^{-1}u^\dagger(\xi,\eta,\lambda^*)A_0 = u^{-1}(\xi,\eta,\lambda)$, after taking the limit $\lambda\to\lambda_k$, we obtain algebraic equation for $|n_k\rangle$ in terms of $\langle m_k^T|$

$$|\nu\rangle = \mathcal{M}^{-1}|\mu\rangle. \tag{74}$$

Below, for simplicity, we write down the matrix $\mathcal{M}$ for $N_1 = N_2 = 1$

$$|\nu\rangle = \begin{pmatrix} |n_1\rangle \\ \hline |n_2\rangle \\ |n_2^*\rangle \end{pmatrix}, \quad |\mu\rangle = \begin{pmatrix} A_0|m_1\rangle \\ \hline A_0|m_2\rangle \\ A_0|m_2^*\rangle \end{pmatrix}, \quad \mathcal{M} = \left(\begin{array}{c|cc} A & B & B^* \\ \hline B & D & E \\ -B^* & -E^* & D^* \end{array}\right) \tag{75}$$

$$\begin{aligned}
A &= \frac{1}{2\lambda_1^3}\operatorname{diag}(Q^{(1)},Q^{(2)},Q^{(3)}), & B &= \frac{1}{\lambda_1^3+\lambda_2^3}\operatorname{diag}(P^{(1)},P^{(2)},P^{(3)}) \\
D &= \frac{1}{2\lambda_2^3}\operatorname{diag}(T^{(1)},T^{(2)},T^{(3)}), & E &= \frac{1}{\lambda_2^{*,3}-\lambda_2^3}\operatorname{diag}(K^{(1)},K^{(2)},K^{(3)}) \\
Q^{(j)} &= \langle m_1^T|\Lambda_{11}^{(j)}(\lambda_1,\lambda_1)|m_1\rangle, & P^{(j)} &= \langle m_1^T|\Lambda_{12}^{(j)}(\lambda_1,\lambda_2)|m_2\rangle \\
T^{(j)} &= \langle m_2^T|\Lambda_{22}^{(j)}(\lambda_2,\lambda_2)|m_2\rangle, & K^{(j)} &= \langle m_2^T|\Lambda_{22}^{(j)}(\lambda_2,-\lambda_2^*)|m_2^*\rangle
\end{aligned} \tag{76}$$

with

$$\Lambda_{lp}^{(j)} = -\lambda_l\lambda_p E_{3-j,1+j} + \lambda_l^2 E_{2-j,2+j} + \lambda_p^2 E_{1-j,3+j}, \qquad j = 1,2,3. \tag{77}$$

In order to obtain the two-soliton solution of the Tzitzeica equation we take the limit $\lambda\to 0$ in the equations satisfied by the dressing factor $u(\xi,\eta,\lambda)$ and integrate to get

$$\phi_{\mathrm{Ns}}(\xi,\eta) = -\frac{1}{2}\ln\left|1 - \frac{n_{1,1}m_{1,1}}{\lambda_1} - \frac{n_{2,1}m_{2,1}}{\lambda_2} - \frac{n_{2,1}^*m_{2,1}^*}{\lambda_2^*}\right|. \tag{78}$$

The above formulae can be easily generalized for any N_1 and N_2.

5. Hirota Method for Building One-soliton Solution of T2 Equation

There are many methods for deriving the soliton solutions and we have demonstrated two of the most used: the dressing method and the Hirota method [3, 6, 17]. Both methods give the same results both for the kinks and for the breathers.

We build the Hirota bilinear form of T2 equation using the substitution

$$\phi(\xi,\eta) = \frac{1}{2}\ln\frac{g(\xi,\eta)}{f(\xi,\eta)}. \tag{79}$$

Introducing it into the second equation in (1) and decoupling in the bilinear dispersion relation and the soliton-phase constraint, we obtain the following system

$$\frac{\partial^2 g}{\partial\xi\partial\eta}g - \frac{\partial g}{\partial\xi}\frac{\partial g}{\partial\eta} - f^2 + g^2 = 0, \qquad \frac{\partial^2 f}{\partial\xi\partial\eta}g - \frac{\partial f}{\partial\xi}\frac{\partial f}{\partial\eta} - fg + f^2 = 0. \tag{80}$$

We impose that

$$g(\xi,\eta) = 1 + az(\xi,\eta) + bz^2(\xi,\eta), \quad f(\xi,\eta) = 1 + Az(\xi,\eta) + Bz^2(\xi,\eta) \tag{81}$$

where $z(\xi,\eta) = \mathrm{e}^{k\xi-\omega\eta}$, k - the wave number, ω - the angular frequency.

Using a software for analytical computation like `Mathematica`, we obtain that

$$\begin{aligned} g(\xi,\eta) &= 1 - 2A\mathrm{e}^{k\xi-\frac{3}{k}\eta} + \frac{A^2}{4}\mathrm{e}^{k\xi-\frac{3}{k}\eta} \\ f(\xi,\eta) &= 1 + A\mathrm{e}^{k\xi-\frac{3}{k}\eta} + \frac{A^2}{4}\mathrm{e}^{k\xi-\frac{3}{k}\eta} \end{aligned} \tag{82}$$

where the dispersion relation is $\omega = \frac{3}{k}\cdot$

Using the above results our one-soliton solution for T2 acquires the following form

$$\phi(\xi,\eta) = \frac{1}{2}\ln\left[\frac{3}{2}\tanh^2\left(\frac{1}{2}(k\xi - \frac{3}{k}\eta)\right) - \frac{1}{2}\right]. \tag{83}$$

This solution coincide with the one obtained by Mikhailov in [21] for $k = \sqrt{3}\rho_1$. In this very direct manner, Hirota method gives immediately the one-soliton solution of the first type, which we have obtained also in (49) through the dressing method, as a particular case of a more general form (46).

One can also use the standard Hirota technique to derive N-soliton solution of first type each parametrized with real eigenvalue ρ_k and a vector $(\mu_{k,1}, \mu_{k,2}, \mu_{k,3})$ with $\mu_{k,2} = 0$. We believe, that using Hirota method one can derive also more complicated cases of one- and N-soliton solutions. To this end, however one needs a more complicated ansatz for the functions $f(\xi,\eta)$ and $g(\xi,\eta)$ which would solve equation (80) but could not be reduced to functions of $z(\xi,\eta)$ only.

Of course, the equation (80) can be solved in a more general case, but the only one solution we were able to obtain by now, using the well known ansatz (81), was (82), which corresponds to the soliton solutions of first type. To find $g(\xi,\eta)$ and $f(\xi,\eta)$ corresponding to the second type of soliton solutions is still an open problem for us and it will be tackled in a next paper. A possible approach could be to start from the second type solitons given by the dressing factor method and, on this basis, to guess the ansatz which should be imposed to obtain $g(\xi,\eta)$ and $f(\xi,\eta)$ verifying (80).

6. The Spectral Properties of the Dressed Lax Operator

Here we shall demonstrate that each dressing adds to the discrete spectrum of L sets of discrete eigenvalues.

In our previous paper we showed that the Lax operator has a set of 6 fundamental analytic solutions. We will denote them by $\chi_\nu(\xi,\eta,\lambda)$ where $\nu=0,\dots,5$ denotes the number of the sector $\Omega_\nu \equiv \frac{(2\nu+1)\pi}{6} \le \arg\lambda \le \frac{(2\nu+3)\pi}{6}$, i.e., those are the sectors closed by the rays $(l_\nu, l_{\nu+1})$.

The dressing factor for solitons of first type (23) obviously has simple poles located at $|\lambda_1|\mathrm{e}^{2\pi\mathrm{i}\,k/3}$, $k=0,1,2$. The inverse of this dressing factor has also simple poles located at $|\lambda_1|\mathrm{e}^{\pi\mathrm{i}\,(2k+1)/3}$, $k=0,1,2$.

Each dressing factor for soliton of second type (58) has 6 simple poles located at $|\lambda_2|\mathrm{e}^{\mathrm{i}\,\beta_1+2\pi\mathrm{i}\,k/3}$ and $|\lambda_2|\mathrm{e}^{-\mathrm{i}\,\beta_1+\pi\mathrm{i}\,(2k+1)/3}$, $k=0,1,2$. The inverse of this dressing factor has also 6 simple poles located at $|\lambda_2|\mathrm{e}^{\mathrm{i}\,\beta_1+\pi\mathrm{i}\,(2k+1)/3}$ and $|\lambda_2|\mathrm{e}^{-\mathrm{i}\,\beta_1+2\pi\mathrm{i}\,k/3}$, $k=0,1,2$.

The FAS can be used to construct the kernel of the resolvent of the Lax operator L. In this section by $\chi^\nu(\xi,\lambda)$ we will denote

$$\chi^\nu(\xi,\lambda) = u(\xi,\lambda)\chi_0^\nu(\xi,\lambda)u_-^{-1}(\lambda), \qquad u_-(\lambda) = \lim_{\xi\to-\infty} u(\xi,\eta,\lambda) \tag{84}$$

where $\chi_0^\nu(\xi,\lambda)$ is a regular FAS and $u(\xi,\lambda)$ is a dressing factor of general form (72). Let us introduce

$$\begin{aligned} R^\nu(\xi,\xi',\lambda) &= \frac{1}{\mathrm{i}}\chi^\nu(\xi,\lambda)\Theta_\nu(\xi-\xi')\hat{\chi}^\nu(\xi',\lambda) \\ \Theta_\nu(\xi-\xi') &= \mathrm{diag}\,\Big(\eta_\nu^{(1)}\theta(\eta_\nu^{(1)}(\xi-\xi')), \eta_\nu^{(2)}\theta(\eta_\nu^{(2)}(\xi-\xi')), \eta_\nu^{(3)}\theta(\eta_\nu^{(3)}(\xi-\xi'))\Big) \end{aligned} \tag{85}$$

where $\theta(\xi-\xi')$ is the step-function and $\eta_\nu^{(k)}=\pm1$, see the Table 2.

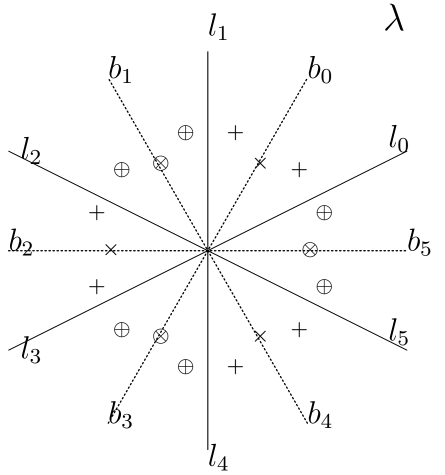

Figure 1. The contour of the RHP with $\mathbb{Z}_3$-symmetry fills up the rays $l_0, \ldots, l_5$. The symbols $\times$ and $\otimes$ (respectively $+$ and $\oplus$) denote the locations of the discrete eigenvalues corresponding to a soliton of first (respectively second) type.

Table 2. The set of signs $\eta_\nu^{(k)}$ for each of the sectors Υ_ν (86).

	Υ_0	Υ_1	Υ_2	Υ_3	Υ_4	Υ_5
$\eta_\nu^{(1)}$	$-$	$-$	$-$	$+$	$+$	$+$
$\eta_\nu^{(2)}$	$+$	$+$	$-$	$-$	$-$	$+$
$\eta_\nu^{(3)}$	$-$	$+$	$+$	$+$	$-$	$-$

Then the following theorem holds true [4]

Theorem 2. *Let $Q(\xi)$ be a Schwartz-type function and let $\lambda_j^\pm$ be the simple zeroes of the dressing factor $u(\xi, \lambda)$ (72). Then*

1. *The functions $R^\nu(\xi, \xi', \lambda)$ are analytic for $\lambda \in \Upsilon_\nu$ where*

$$b_\nu\colon\ \arg\lambda = \frac{\pi(\nu+1)}{3}, \qquad \Upsilon_\nu\colon\ \frac{\pi(\nu+1)}{3} \leq \arg\lambda \leq \frac{\pi(\nu+2)}{3} \tag{86}$$

   having pole singularities at $\pm\lambda_j^\pm$

2. *$R^\nu(\xi, \xi', \lambda)$ is a kernel of a bounded integral operator for $\lambda \in \Upsilon_\nu$*

3. $R^\nu(\xi,\xi',\lambda)$ *is uniformly bounded function for* $\lambda \in b_\nu$ *and provides a kernel of an unbounded integral operator*

4. $R^\nu(\xi,\xi',\lambda)$ *satisfy the equation*

$$L(\lambda)R^\nu(\xi,\xi',\lambda) = \mathbb{1}\delta(\xi-\xi'). \tag{87}$$

Remark 3. *The dressing factor* $u(\xi,\lambda)$ *has* $3N_1+6N_2$ *simple poles located at* $\lambda_l q^p$, $\lambda_r q^p$ *and* $\lambda_r^* q^p$ *where* $l = 1,\dots,N_1$, $r = 1,\dots,N_2$ *and* $p = 0,1,2$. *Its inverse* $u^{-1}(\xi,\lambda)$ *has also* $3N_1+6N_2$ *poles located* $-\lambda_l q^p$, $-\lambda_r q^p$ *and* $-\lambda_r^* q^p$. *In what follows for brevity we will denote them by* λ_j, $-\lambda_j$ *for* $j = 1,\dots,3N_1+6N_2$.

It remains to show that the poles of $R^\nu(\xi,\xi',\lambda)$ coincide with the poles of the dressing factors $u(\xi,\lambda)$ and its inverse $u^{-1}(\xi,\lambda)$.

The proof follows immediately from the definition of $R^\nu(\xi,\xi',\lambda)$ and from equation (84), taking into account that the limiting value $u_-(\lambda)$ commutes with the corresponding matrix $\Theta_\nu(\xi-\xi')$.

Thus we have established that dressing by the factor $u(\xi,\lambda)$, we in fact add to the discrete spectrum of the Lax operator $6N_1+12N_2$ discrete eigenvalues. For $N_1 = N_2 = 1$ they are shown on Figure 1.

7. Conclusions

Shortly before finishing this paper we became aware of the fact, that appropriate combination of changes of variables, considered in Section 2 can take each member of Tzitzeica family (2) into one of its four versions that we introduced. Let us demonstrate how this can be done for the equation

$$\frac{\partial^2\phi}{\partial\xi\partial\eta} = -c_1^2 \mathrm{e}^{2\phi} + c_2^2 \mathrm{e}^{-4\phi} \tag{88}$$

where c_1 and c_2 are real positive constants. Now we shall use somewhat more general change of variables. First we apply the transformation (8) with $s_0 = 0$ and $\phi' = \phi + \ln(c_1/c_2)$. Then we change $\xi \to \xi'/k$, $\eta \to \eta'/k$ where k is also real positive constant taken to be $k = \sqrt[3]{c_1^2 c_2}$. Easy calculation shows that as a result equation (88) goes into T2 for $\phi'(\xi,\eta)$. Using in addition Table 1 we can transform each member of Tzitzeica family into T2 and then solve it using the results above.

Let us consider the soliton solutions Tzitzeica equation in a small neighborhood around the singularities, where $\phi_{\mathrm{as}}(\xi,\eta)$ tends to ∞. Then the second term in the

T2 equation can be neglected and the asymptotically we get

$$2\frac{\partial\phi_{\mathrm{as}}}{\partial\xi\partial\eta} = \mathrm{e}^{2\phi_{\mathrm{as}}}.$$

Similarly if in a small neighborhood around the singularity $\phi'_{\mathrm{as}}(\xi,\eta)$ tends to $-\infty$ we have

$$2\frac{\partial\phi'_{\mathrm{as}}}{\partial\xi\partial\eta} = -\mathrm{e}^{-4\phi'_{\mathrm{as}}}.$$

In both cases we find equations, equivalent to the Liouville equation. Thus the asymptotical behavior of the solutions of Tzitzeica equation around the singularities must be the same as the singularities of Liouville equation [26].

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