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RELATIONS AMONG LOW-DIMENSIONAL SIMPLE LIE GROUPS

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Abstract. The compact classical Lie groups can be regarded as groups of $n \times n$ matrices over the real, complex, and quaternion fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ that satisfy metric- and volume-conserving conditions. These groups, $\mathrm{SO}(n, \mathbb{R}), \mathrm{SU}(n, \mathbb{C})$, and $\mathrm{Sp}(n, \mathbb{Q})$, are not all independent. Homomorphisms exist among some of these groups for small dimension. We review these relations by describing the Lie algebras of the compact forms and their complex extensions. Other noncompact real forms of these Lie algebras are constructed by systematic methods. The relations among all distinct real forms is presented.

Contents

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1. Introduction

A number of isomorphisms and homomorphisms exist among the low-dimensional Lie groups. These relations are classical – they have been known for a long time [9]. The best-known of these is the $2:1$ relation between the unitary group $\mathrm{SU}(2)$ and the orthogonal group $\mathrm{SO}(3)$ [57]. In one direction all irreducible representations of $\mathrm{SU}(2)$ are tensor representations of degree $2j = 0, 1, 2, \cdots$ and dimension $2j+1$. Those with j integer are also tensor representations of $\mathrm{SO}(3)$. In the reverse direction, $\mathrm{SO}(3)$ has irreducible tensor representations of degree l and dimension $2l+1$. There are also spinor "representations" of $\mathrm{SO}(3)$. These are in fact representations of the covering group $\widetilde{\mathrm{SO}(3)} \simeq \mathrm{SU}(2)$ of $\mathrm{SO}(3)$. These spin representations were mysterious to physicists until the relation between the two compact Lie groups $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ was well-understood [56, 57].

There are a number of other isomorphisms and homomorphisms among simple Lie groups of low dimension. These are most easily explored in terms of the relations among their Lie algebras. In this work we review the relations among the low-dimensional complex Lie algebras. This comparison is facilitated by identifying the low-dimensional groups and algebras by their dimension and rank [37,56]. Among the simple complex Lie algebras we find $A_1 = B_1 = C_1$, $B_2 = C_2$, and $D_3 = A_3$. For convenience, we include also the semisimple case $D_2 = A_1 \oplus A_1$. The real forms of all these complex Lie algebras are identified and related to each other. We also include a list, for each, of the fundamental irreducible representations and discuss the existence of "spinor" representations for the orthogonal groups: $\mathrm{SO}(n)$, $n = 3, 4, 5, 6$. The spinor representations are fundamental representations of these orthogonal algebras and defining representations (or their complex conjugates) of real forms of related Lie algebras: for example $\mathrm{SU}(4) \simeq A_3$ has two inequivalent four dimensional fundamental representations that are complex conjugates of each other. Therefore $\mathrm{SO}(6) \simeq D_3$ has two inequivalent four-dimensional spinor representations that are complex conjugates of each other.

2. Classical Results

The complete list of all simple Lie algebras over the complex field consists of four infinite series A_n, B_n, C_n, D_n, together with five exceptional algebras: G_2, F_4, E_6, E_7, E_8 [9, 26, 37, 56]. More accurately, these structures are root spaces: sets of vectors in Euclidean spaces $\mathbb{R}^n$ whose roots describe the commutation relations of the basis vectors in the Lie algebra. The classical algebras are closely associated with the algebras of classical matrix groups as shown in Table 1. The subscript n identifies the *rank* of the algebra: this is the maximum number of independent simultaneously commuting operators that can be constructed in the algebra. The unitary groups $\mathrm{U}(n)$ are defined by a metric-preserving condition $\mathrm{U}^\dagger \mathrm{I}_n \mathrm{U} = \mathrm{I}_n$ and their special unitary subgroups $\mathrm{SU}(n)$ satisfy in addition the volume-preserving condition $\det(\mathrm{U}) = +1$. The orthogonal matrix groups $\mathrm{O}(n)$ are real subgroups of the complex matrix groups $\mathrm{U}(n)$ and the special subgroups $\mathrm{SO}(n) \subset \mathrm{SU}(n)$ which obey in addition the volume-preserving condition. The symplectic groups have been defined in a number of ways. Following in the footsteps of the special orthogonal and unitary groups $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$, which are compact groups of $n \times n$ matrices over the real and complex fields, we can define the symplectic group $\mathrm{Sp}(n, \mathbb{Q})$ of $n \times n$ quaternion-valued matrices that obey a metric-preserving condition as well as a volume-conserving condition. This group is compact. Alternatively, we can define symplectic groups $\mathrm{Sp}(2n, \mathbb{R})$ as real $2n \times 2n$ matrix groups that preserve a real nonsingular antisymmetric matrix A, such as occurs in

Table 1. Classical Lie groups associated with simple matrix groups with their dimensions and defining conditions. Lie algebras associated with these Lie groups are denoted $\mathfrak{su}(n), \mathfrak{so}(2n), \mathfrak{so}(2n+1), \mathfrak{sp}(2n, \mathbb{R})$, respectively.

Root Space	Classical Matrix Group	Dimension	Defining Conditions	
A_{n-1}	$\mathrm{SU}(n)$	n^2-1	$\mathrm{U}^\dagger \mathrm{I}_n \mathrm{U} = \mathrm{I}_n$	$\det \mathrm{U} = 1$
D_n	$\mathrm{SO}(2n)$	$n(2n-1)$	$\mathrm{O}^t \mathrm{I}_{2n} \mathrm{O} = \mathrm{I}_{2n}$	$\det \mathrm{O} = 1$
B_n	$\mathrm{SO}(2n+1)$	$n(2n+1)$	$\mathrm{O}^t \mathrm{I}_{2n+1} \mathrm{O} = \mathrm{I}_{2n+1}$	$\det \mathrm{O} = 1$
C_n	$\mathrm{Sp}(2n, \mathbb{R})$	$n(2n+1)$	$\mathrm{M}^t \mathrm{A} \mathrm{M} = \mathrm{A}$	$\det \mathrm{M} = 1$

the Hamilton-Jacobi equations of motion, viz: $\begin{bmatrix} 0_n & \mathrm{I}_n \\ -\mathrm{I}_n & 0_n \end{bmatrix}$. This matrix group is noncompact. The compact group $\mathrm{Sp}(n, \mathbb{Q})$ and the noncompact group $\mathrm{Sp}(2n, \mathbb{R})$ have the same complex extension Lie algebra whose root space is C_n [26].

A Lie algebra is first and foremost a linear vector space on which an additional compositional structure exists. It is useful to choose a convenient set of basis vectors in this space. For simple algebras of rank n, it is possible to find n basis operators H_i, $1 \leq i \leq n$ that mutually commute, i.e., $[H_i, H_j] = 0$. These operators can be chosen to be simultaneously diagonal in any matrix representation. Each of the remaining basis vectors (*shift operators*) is associated with a root $\boldsymbol{\alpha}$ of a secular equation: $E_{\boldsymbol{\alpha}}$. The roots $\boldsymbol{\alpha}$ are vectors in the *root space*, which is an n-dimensional linear vector space with Euclidean metric. The commutators of the diagonal operators with the shift operators have the form [9, 37, 56]

$$[H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}} \, . \tag{1}$$

In other words, under commutation the shift operators are eigenoperators of the diagonal operators.

The nonzero roots for the classical Lie algebras can be chosen in the following convenient way in terms of a set of vectors $\mathbf{e}_k$ that are orthonormal ($\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$) [26, 37]

$$\begin{array}{lllll} A_{n-1} & \pm(\mathbf{e}_i - \mathbf{e}_j), & & 1 \leq i < j \leq n & \\ D_n & \pm\mathbf{e}_i \pm \mathbf{e}_j, & & 1 \leq i < j \leq n & \\ B_n & \pm\mathbf{e}_i \pm \mathbf{e}_j, & \pm\mathbf{e}_k, & 1 \leq i < j \leq n, & 1 \leq k \leq n \\ C_n & \pm\mathbf{e}_i \pm \mathbf{e}_j, & \pm 2\mathbf{e}_k, & 1 \leq i < j \leq n, & 1 \leq k \leq n. \end{array} \tag{2}$$

Within each of these vector spaces it is convenient to choose a set of *natural* basis vectors. The choice for rank-three Lie algebras is [17]

$$\begin{array}{llll} A_3 & \mathbf{e}_1 - \mathbf{e}_2, & \mathbf{e}_2 - \mathbf{e}_3, & \mathbf{e}_3 - \mathbf{e}_4 \\ D_3 & \mathbf{e}_1 - \mathbf{e}_2, & \mathbf{e}_2 - \mathbf{e}_3, & \mathbf{e}_2 + \mathbf{e}_3 \\ B_3 & \mathbf{e}_1 - \mathbf{e}_2, & \mathbf{e}_2 - \mathbf{e}_3, & 1\mathbf{e}_3 \\ C_3 & \mathbf{e}_1 - \mathbf{e}_2, & \mathbf{e}_2 - \mathbf{e}_3, & 2\mathbf{e}_3. \end{array} \tag{3}$$

The restriction to lower-rank algebras, or extension to higher-rank classical algebras, is straightforward. This choice has two virtues

Lie algebra: Every nonzero root for a Lie algebra can be expressed as a linear combination of the natural basis roots with integer coefficients. Further, the nonzero integers in this decomposition are either all positive (for *positive roots*) or all negative [17, 37].

Representations: Every tensor product representation of the Lie algebra/group can be expressed in terms of integers related to this choice of basis [56].

3. Relations Among Low-Dimensional Algebras

Several equivalences exist among the low-rank Lie algebras [26, 37]. Two algebras can be equivalent only if they have the same rank and dimension. All rank-one (complex) algebras are equivalent: $A_1 = B_1 = C_1$. A rank-two orthogonal and the symplectic algebra are equivalent: $B_2 = C_2$ and a rank-three orthogonal algebra and the unitary algebra are equivalent: $D_3 = A_3$ [26, 37]. The rank-two orthogonal algebra D_2 has four nonzero roots $\pm\mathbf{e}_1 \pm \mathbf{e}_2$, which split into two mutually orthogonal pairs, indicating that D_2 is the direct sum of two simple rank-one Lie algebras: $D_2 \simeq A_1 \oplus A_1$. D_2 is not simple: it is semisimple. These equivalences are summarized in Table 2.

4. Real Forms

An element in a complex Lie algebra has the general form $\sum_i^n h^i H_i + \sum_{\boldsymbol{\alpha}\neq 0} \mathrm{e}^{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}}$, where the coefficients h^i and $\mathrm{e}^{\boldsymbol{\alpha}}$ are complex [26, 29, 37]. Complex Lie algebras have several different real forms. They are obtained by imposing specific reality conditions on these coefficients. It is always possible to find a real form that is compact: it is the Lie algebra for a compact group. The compact groups are the orthogonal groups $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n+1)$ (D_n and B_n series), the unitary groups $\mathrm{SU}(n)$

Table 2. Equivalences among the classical complex Lie algebras. D_2 is semisimple. There is a $1:1$ correspondence between root spaces and complex simple Lie algebras.

Rank	Algebras
1	$A_1 = B_1 = C_1$
2	$D_2 = A_1 \oplus A_1$
2	$B_2 = C_2$
3	$A_3 = D_3$

(A_{n-1}) and the corresponding matrix groups over the quaternion field $\mathrm{Sp}(n, \mathbb{Q})$ (C_n). Another inequivalent real form can also always be constructed. It is the *least compact* real form, obtained by restricting the complex coefficients h^i and $\mathrm{e}^{\boldsymbol{\alpha}}$ to be real. The distinct real forms are distinguished by an index, χ. This is the difference of the dimensions of two subspaces in the real form, $\chi = d_{n.c.} - d_c$, where d_c is the dimension of the maximal compact subspace in the real form of the algebra and $d_{n.c.}$ is the dimension of its complement, which is noncompact. Since the dimension of a real Lie algebra is $\dim(\mathfrak{g}) = d_{n.c.} + d_c$, $\chi(\mathfrak{g}) = 2d_{n.c.} - \dim(\mathfrak{g})$.

Cartan transformed the search for real forms to a simple and elegant linear vector space problem – in fact, a simple problem of matrices [9, 37]. He searched for a mapping T of the algebra to itself that obeyed $T^2 = \mathrm{Id}$ (*involutive automorphism*). Under such an automorphism the linear vector space $\mathfrak{g}$ splits into two eigenspaces, one associated with each of the eigenvalues of T

$$\begin{array}{rcl} \mathfrak{g} &=& \mathfrak{k} + \mathfrak{p} \\ T(\mathfrak{k}) &=& +\mathfrak{k} \\ T(\mathfrak{p}) &=& -\mathfrak{p} \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} [\mathfrak{k}, \mathfrak{k}] &\subseteq& \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] &=& \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] &\subseteq& \mathfrak{k}\,. \end{array} \tag{4}$$

If $\mathfrak{p} \neq 0$, the substitution $\mathfrak{p} \to \mathfrak{p}' = \mathrm{i}\mathfrak{p}$ maps the original real form to a different real form.

If $\mathfrak{g}$ is the Lie algebra for a compact Lie group G, then $\mathfrak{k}$ and $\mathfrak{p}$ are compact, $\mathfrak{p}'$ is noncompact, and $\chi(\mathfrak{g}') = \dim(\mathfrak{p}) - \dim(\mathfrak{k})$. Starting from the maximal compact Lie algebra $\mathfrak{g}$ it is then possible, by using one of three types of involutive automorphisms, to construct all possible real forms of all the simple (and semisimple) classical Lie algebras. The three types of involutive automorphisms are summarized here [26, 29, 37]. They will be applied to the compact real forms of the matrices that define the Lie groups and their algebras. The character of a real form ranges from $-\dim\,(\mathfrak{g})$ for the compact real form to $+\mathrm{rank}\,(\mathfrak{g})$ for the least compact real form of a complex Lie algebra.

4.1. Block Matrix Decomposition

The matrix Lie algebras for the special orthogonal, special unitary, and symplectic groups $\mathrm{SO}(n,\mathbb{R}), \mathrm{SU}(n,\mathbb{C})$, and $\mathrm{Sp}(n,\mathbb{Q})$ have the form of traceless antihermitian matrices over the real, complex, and quaternion fields [26,29]. Define the diagonal matrix $\mathrm{I}_{p,q} = \begin{bmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{bmatrix}$ with $p+q=n$. Then the block diagonal involution is the mapping $\mathfrak{g} \to \mathrm{I}_{p,q}\ \mathfrak{g}\ \mathrm{I}_{p,q}$. Under this involution we find

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \left[\begin{array}{c|c} A_p & B \\ \hline -B^\dagger & A_q \end{array}\right] \xrightarrow{T=\mathrm{I}_{p,q}} \left[\begin{array}{c|c} A_p & B \\ \hline +B^\dagger & A_q \end{array}\right] = \mathfrak{k} + \mathrm{i}\mathfrak{p} = \mathfrak{k} + \mathfrak{p}' = \mathfrak{g}'. \tag{5}$$

For the compact real form the block diagonal submatrices A_p, A_q are antihermitian $A_p^\dagger = -A_p$ and $\mathrm{tr}\ (A_p + A_q) = 0$. The dimension of the noncompact subspace of $\mathfrak{g}'$ is $d_{n.c.} = pq \times \dim F$, where $\dim F = 1,2,4$ for the real, complex, and quaternion fields. The characters of the noncompact real forms are

$$\begin{aligned} \chi(\mathfrak{so}(p,q)) &= \tfrac{1}{2}\left[(p+q) - (p-q)^2\right] \\ \chi(\mathfrak{su}(p,q)) &= 1 - (p-q)^2 \\ \chi(\mathfrak{sp}(p,q;Q)) &= -(p+q) - 2(p-q)^2 \ . \end{aligned} \tag{6}$$

4.2. Subfield Restriction

Matrices that describe the Lie algebra $\mathfrak{su}(n)$ are traceless and antihermitian and can be expressed as the sum of two $n \times n$ matrices: a real antisymmetric matrix $A = -A^t$ and i times a real symmetric traceless matrix $B = B^t$

$$\mathfrak{su}(n) = A + \mathrm{i}B \ . \tag{7}$$

Under complex conjugation $A \to +A$ and $\mathrm{i}B \to -\mathrm{i}B$. The image of $\mathfrak{su}(n)$ under this involution is the real matrix $A+B$. This describes the Lie algebra of the real special linear group $\mathfrak{sl}(n,\mathbb{R})$. The dimension of the noncompact subspace in this algebra is $d_{n.c.} = n(n+1)/2 - 1$.

A similar decomposition can be carried out on the symplectic group $\mathrm{Sp}(n,\mathbb{Q})$. It has a matrix representation in terms of quaternion-valued traceless antihermitian $n \times n$ matrices. We can perform the same type of involution using quaternions as done above using complex numbers under the decomposition [26,29]

$$q_0 + q_1\mathcal{I} + q_2\mathcal{J} + q_3\mathcal{K} \xrightarrow{T} q_0 + q_1\mathcal{I} - q_2\mathcal{J} - q_3\mathcal{K} \ . \tag{8}$$

However, an alternative procedure is more economical. First we replace each quaternion by a 2×2 complex matrix

$$q_0 + q_1\mathcal{I} + q_2\mathcal{J} + q_3\mathcal{K} \to \begin{bmatrix} q_0 + \mathrm{i}q_3 & \mathrm{i}q_1 + q_2 \\ \mathrm{i}q_1 - q_2 & q_0 - \mathrm{i}q_3 \end{bmatrix} . \tag{9}$$

When each quaternion in $\mathfrak{sp}(n, \mathbb{Q})$ is replaced by a 2×2 complex matrix we obtain a complex $2n \times 2n$ matrix

$$\mathfrak{sp}(n, \mathbb{Q}) \xrightarrow{\text{equation (9)}} \mathfrak{ou}(2n, \mathbb{R}) + [\mathfrak{usp}(2n, \mathbb{C}) - \mathfrak{ou}(2n, \mathbb{R})] \ . \tag{10}$$

In this expression $\mathfrak{ou}(2n, \mathbb{R})$ is a $2n \times 2n$ real matrix representation of $n \times n$ unitary matrices $\mathfrak{u}(n, \mathbb{C})$, and $\mathfrak{usp}(2n, \mathbb{C})$ is a $2n \times 2n$ complex representation of the quaternion valued $n \times n$ matrix $\mathfrak{sp}(n, \mathbb{Q})$. Now the involution used above in equation (7) is used to map $\mathfrak{sp}(n, \mathbb{Q}) = \mathfrak{usp}(2n, \mathbb{C})$ to its related noncompact form $\mathfrak{sp}(2n, \mathbb{R})$.

The characters of the noncompact real forms obtained through this involution are

$$\chi(\mathfrak{sl}(n, \mathbb{R})) = n - 1, \qquad \chi(\mathfrak{sp}(2n, \mathbb{R})) = n. \tag{11}$$

These real forms are the least compact real forms of the complex Lie algebra since these values are the rank of the complex extension algebras A_{n-1}, C_n.

4.3. Field Embeddings

The third useful matrix involution involves the reverse process. The Lie algebra $\mathfrak{so}(2n)$ of antisymmetric matrices contains as a subalgebra the set of matrices $\mathfrak{ou}(2n)$, obtained by representing the antihermitian $n \times n$ matrices of $\mathfrak{u}(n, \mathbb{C})$ as real $2n \times 2n$ matrices. This leads to a simple decomposition

$$\begin{aligned} \mathfrak{so}(2n) &= \mathfrak{ou}(2n) + [\mathfrak{so}(2n) - \mathfrak{ou}(2n)] \\ &\to \mathfrak{ou}(2n) + \mathrm{i}\,[\mathfrak{so}(2n) - \mathfrak{ou}(2n)] = \mathfrak{so}^*(2n). \end{aligned} \tag{12}$$

The Cartan process is followed by multiplying the complement of $\mathfrak{ou}(2n)$ by i, leading to a real form labeled $\mathfrak{so}^*(2n)$.

Following the same procedure we find

$$\begin{aligned} \mathfrak{su}(2n) &= \mathfrak{usp}(2n, \mathbb{C}) + [\mathfrak{su}(2n) - \mathfrak{usp}(2n, \mathbb{C})] \\ &\to \mathfrak{usp}(2n, \mathbb{C}) + \mathrm{i}\,[\mathfrak{su}(2n) - \mathfrak{usp}(2n, \mathbb{C})] = \mathfrak{su}^*(2n). \end{aligned} \tag{13}$$

The characters of the noncompact real forms obtained through this involution are [26]

$$\chi(\mathfrak{so}^*(2n)) = -n, \qquad \chi(\mathfrak{su}^*(2n)) = -2n - 1. \tag{14}$$

Table 3. Equivalences among the real forms of the rank-one Lie algebras $B_1 = A_1 = C_1$.

B_1	A_1	C_1	χ
$\mathfrak{so}(3)$	$\mathfrak{su}(2)$	$\mathfrak{sp}(1,\mathbb{Q})$	-3
$\mathfrak{so}(2,1)$	$\mathfrak{su}(1,1) = \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sp}(2,\mathbb{R})$	1

Table 4. Equivalences among the real forms of the rank-two Lie algebras $D_2 = A_1 \oplus A_1$.

D_2	$A_1 \oplus A_1$	χ
$\mathfrak{so}(4)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	-6
$\mathfrak{so}^*(4)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(1,1)$	-2
$\mathfrak{so}(3,1)$	$\mathfrak{sl}(2,c)$	0
$\mathfrak{so}(2,2)$	$\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$	2

5. Equivalences of Real Forms

The complex Lie algebras A_1, B_1, C_1 are equivalent, so their various real forms must be equivalent. The equivalences are shown in Table 3. The equivalences among the real forms of the semisimple Lie algebra D_2 and the direct sum $A_1 \oplus A_1$ are shown in Table 4. In the same way, the equivalences among the real forms of $B_2 = C_2$ are shown in Table 5, while those for $D_3 = A_3$ are shown in Table 6.

6. Fundamental Representations

Every simple Lie algebra of rank n has n fundamental irreducible representations [17, 56]. Every irreducible representation of the Lie algebra can be constructed as a tensor product from these fundamental irreducible representations. These representations are defined by their highest weights. These weights have the

Table 5. Equivalences among the real forms of the rank-two Lie algebras B_2 and C_2.

B_2	C_2	χ
$\mathfrak{so}(5)$	$\mathfrak{sp}(2,\mathbb{Q}) = \mathfrak{usp}(4,\mathbb{C})$	-10
$\mathfrak{so}(4,1)$	$\mathfrak{sp}(1,1;\mathbb{Q})$	-2
$\mathfrak{so}(3,2)$	$\mathfrak{sp}(4,\mathbb{R})$	2

Table 6. Equivalences among the real forms of the rank-two Lie algebras D_3 and A_3.

D_3	A_3	χ
$\mathfrak{so}(6)$	$\mathfrak{su}(4)$	-15
$\mathfrak{so}(5,1)$	$\mathfrak{su}^*(4)$	-5
$\mathfrak{so}^*(6)$	$\mathfrak{su}(3,1)$	-3
$\mathfrak{so}(4,2)$	$\mathfrak{su}(2,2)$	1
$\mathfrak{so}(3,3)$	$\mathfrak{sl}(4,\mathbb{R})$	3

Table 7. Highest weights $\mathbf{F}$ for the fundamental irreducible representations of low-rank Lie algebras. For A_3 the abbreviation (a,b,c) is shorthand for $a(\mathbf{e}_1 - \mathbf{e}_2) + \cdots$. The dimensions of these representations are listed in the appropriate column. In the column Topology, $s.c.$ indicates the exponential of the representation is simply connected and 2 indicates it is doubly connected.

Algebra	Natural $\mathbf{r}_i$	Fundamental $\mathbf{F}_i$	Dimension	Topology
A_1	$\mathbf{e}_1 - \mathbf{e}_2$	$\frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2)$	2	$s.c.$
B_1	$\mathbf{e}_1$	$\frac{1}{2}\mathbf{e}_1$	2	$s.c.$
C_1	$2\mathbf{e}_1$	$\mathbf{e}_1$	2	$s.c.$
D_2	$\mathbf{e}_1$	$\frac{1}{2}\mathbf{e}_1$	2	$s.c.$
	$\mathbf{e}_2$	$\frac{1}{2}\mathbf{e}_2$	2	$s.c.$
B_2	$\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_1$	5	2
	$\mathbf{e}_2$	$\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$	4	$s.c.$
C_2	$\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_1$	4	$s.c.$
	$2\mathbf{e}_2$	$\mathbf{e}_1 + \mathbf{e}_2$	5	2
D_3	$\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_1$	6	2
	$\mathbf{e}_2 - \mathbf{e}_3$	$\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)$	4	$s.c.$
	$\mathbf{e}_2 + \mathbf{e}_3$	$\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$	4	$s.c.$
A_3	$\mathbf{e}_1 - \mathbf{e}_2$	$\frac{1}{4}(3,2,1)$	4	$s.c.$
	$\mathbf{e}_2 - \mathbf{e}_3$	$\frac{1}{4}(2,4,2)$	6	$s.c.$
	$\mathbf{e}_3 - \mathbf{e}_4$	$\frac{1}{4}(1,2,3)$	4	$s.c.$

following orthonormality properties with respect to the n fundamental roots of the root space [17]

$$2\,\frac{(\mathbf{F}_i, \mathbf{r}_j)}{(\mathbf{r}_j, \mathbf{r}_j)} = \delta_{ij} \tag{15}$$

where $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ are the n natural basis roots for the Lie algebra (cf equation (3)) and $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$ are the highest weights for the first, second, $\cdots$, n^{th}-fundamental irreducible representation. The highest weights of the fundamental representations of the Lie algebras described above are collected in Table 7.

7. Dimensions of Irreducible Representations

Weyl [56] has given us an expression for the dimensions of the irreducible representations of the simple Lie algebras that is breathtaking in its elegance and simplicity.

First, every irreducible representation can be constructed as a tensor product over powers of the fundamental irreducible representations: $(\mathbf{F}_1)^{\otimes m_1} \otimes \cdots \otimes (\mathbf{F}_n)^{\otimes m_n}$. This tensor product is reducible. It contains many irreducible representations, but the representation with the highest weight in this tensor product has weight $\mathbf{\Lambda}$ given by [17,41,49,56]

$$\mathbf{\Lambda} = \sum_{i=1}^{n} m_i \mathbf{F}_i \ . \tag{16}$$

Conversely, given an irreducible representation of highest weight $\mathbf{\Lambda}$, its composition in terms of the fundamental irreducible representations can be determined using the inner product properties of the natural basis vectors of the algebra with the highest weights of the fundamental irreducible representations

$$m_i = 2\frac{(\mathbf{\Lambda}, \mathbf{r}_i)}{(\mathbf{r}_i, \mathbf{r}_i)} \ . \tag{17}$$

Weyl's expression for dimension involves the positive root vectors $\boldsymbol{\alpha}$, half the sum over the positive root vectors $\mathbf{R} = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha}$, and the highest weight $\mathbf{\Lambda} = \sum_i m_i \mathbf{F}_i$

$$\mathrm{Dim}(\mathbf{\Lambda}) = \Pi_{\boldsymbol{\alpha}>0} \frac{(\mathbf{\Lambda} + \mathbf{R}, \boldsymbol{\alpha})}{(\mathbf{0} + \mathbf{R}, \boldsymbol{\alpha})} \ . \tag{18}$$

Example 1. *For the* 15*-dimensional Lie algebra* A_3 *there are six positive roots* $(1,-1,0,0), (0,1,-1,0), (0,0,1,-1), (1,0,-1,0), (1,0,0,-1), (0,1,0,-1)$. *The first three are the natural basis. Half the sum of all positive roots is* $\frac{1}{2}(3,1,-1,-3)$. *The first fundamental irreducible representation has highest weight* $\frac{1}{4}(3,2,1)$ $\rightarrow \frac{1}{4}(3,-1,-1,-1)$. *The dimension is*

$$\mathrm{Dim}(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}) = \frac{2 \times 1 \times 1 \times 3 \times 4 \times 2}{1 \times 1 \times 1 \times 2 \times 3 \times 2} = 4 \ . \tag{19}$$

In this expression the order of the factors corresponds to the order of the positive roots listed above. The dimensions of the other two fundamental representations of A_3 *are* $\mathrm{Dim}(\mathbf{F}_2) = 6$ *and* $\mathrm{Dim}(\mathbf{F}_3) = 4$*. The dimensions of the all fundamental representations for the all low-rank Lie algebras described above are listed in* Table 7.

8. Unitary Irreducible Representations

The compact simple Lie groups all have finite-dimensional unitary irreducible representations that are tensor products based on the set of fundamental irreducible representations [56]. The representations of their Lie algebras consist of complex antihermitian matrices. All the matrix elements for the diagonal and select shift operators in these representations of the compact unitary ($\mathrm{SU}(n)$) [18] and orthogonal ($\mathrm{SO}(2n+1), \mathrm{SO}(2n)$) [19] groups have been written down by Gel'fand and Tsetlein. This calculus has later been extended to representations of the Lie algebras of compact symplectic groups [20,23]. These matrix elements are determined, up to a scale factor, by the branching rules of an irreducible representation under canonical group-subgroup reductions: $\mathrm{SU}(n) \downarrow \mathrm{SU}(n-1) \downarrow \mathrm{SU}(n-2) \cdots$, $\mathrm{SO}(n) \downarrow \mathrm{SO}(n-1) \downarrow \mathrm{SO}(n-2) \downarrow \mathrm{SO}(n-3) \cdots$, and $\mathrm{USp}(2n) \downarrow \cdot \downarrow \mathrm{USp}(2n-2) \downarrow \cdot \downarrow \mathrm{USp}(2n-4) \cdots$ [26,41,49,56]. For the special unitary and orthogonal groups the downarrow ($\downarrow$) means "group-subgroup reduction". For the symplectic groups the double downarrows ($\downarrow \cdot \downarrow$) also indicates a group-subgroup reduction, treated *as if* there were an intermediate group between $\mathrm{USp}(2n)$ and $\mathrm{USp}(2n-2)$. There is not, but the branching rules for irreducible representations act as if that were true [20, 23].

Noncompact real forms are related to compact real forms by analytic continuation. As a result it is possible to construct unitary representations of noncompact real forms by constructing analytic continuations of the Gel'fand-Tsetlein expressions for the matrix elements of the compact real forms. These expressions involve only square root functions of ratios of products of differences. The square root expressions have been treated as *master analytic representations*: functions of the labels that describe basis vectors for the representations in the standard group-subgroup reduction scheme. To extend to noncompact forms, some terms in the difference factors must be replaced to maintain antihermiticity. An example of how this is carried out in the analytic continuation of unitary representations of $\mathrm{SU}(2)$ to unitary representations of $\mathrm{SU}(1,1)$ can be found in [29].

9. Isomorphisms and Homomorphisms

The Lie algebra for the matrix group $\mathrm{SU}(2)$ consists of the Pauli spin matrices $S_j = \frac{\mathrm{i}}{2}\sigma_j$

$$S_1 = \frac{\mathrm{i}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad S_2 = \frac{\mathrm{i}}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{bmatrix}, \qquad S_3 = \frac{\mathrm{i}}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

and the Lie algebra for the matrix group $\mathrm{SO}(3)$ consists of the three 3×3 angular momentum matrices L_j

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad L_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad L_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

These two Lie algebras have isomorphic commutation relations

$$[S_i, S_j] = -\epsilon_{ijk}S_k, \qquad [L_i, L_j] = -\epsilon_{ijk}L_k\ . \tag{22}$$

As a result the algebras are isomorphic and their Lie groups are locally isomorphic. In order to obtain the global relations among these Lie groups we map the algebras to the corresponding groups using the Exponential mapping. This takes the point with (spherical) coordinates $\boldsymbol{\theta} = (\hat{\mathbf{n}}, \theta)$ into the $\mathrm{SU}(2)$ matrix

$$\mathrm{Exp}\left(\frac{\mathrm{i}}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\theta\right) = \cos\frac{\theta}{2}\mathrm{I}_2 + \mathrm{i}\sin\frac{\theta}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} \tag{23}$$

and the same point into the $\mathrm{SO}(3)$ rotation matrix

$$\cos\theta \mathrm{I}_3 + \sin\theta\hat{\mathbf{n}}\cdot\mathbf{L} + (1-\cos\theta)\begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \end{bmatrix}\begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}. \tag{24}$$

This formula, originally due to Rodrigues [50], is simply derived by geometric considerations. A vector $\mathbf{v}$ is decomposed into components parallel and perpendicular to the rotation axis: $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$. The component $\mathbf{v}_{\|}$ is unaffected by the rotation while the perpendicular part $\mathbf{v}_{\perp}$ is rotated through the angle θ.

There is a two-to-one $(2:1)$ relation between these groups. One way to see this is to note that the sphere $(\hat{\mathbf{n}}, 2\pi)$ maps into $-\mathrm{I}_2 \in \mathrm{SU}(2)$ and to $+\mathrm{I}_3 \in \mathrm{SO}(3)$. Two matrices in $\mathrm{SU}(2)$ that differ in sign (i.e., multiplied by $-\mathrm{I}_2$) correspond to a single group operation in $\mathrm{SO}(3)$. Another way to see the global difference between the two Lie groups is to observe that $\mathrm{SU}(2)$ is simply connected: every closed path starting and ending at a point in the parameter space, the Lie algebra, can be continuously deformed to a point. On the other hand $\mathrm{SO}(3)$ is doubly connected. Antipodal points on the surface of a sphere of radius π in the Lie algebra – $(\hat{\mathbf{n}}, \pi)$ and $(-\hat{\mathbf{n}}, \pi)$ – map to the same matrix in $\mathrm{SO}(3)$. It is therefore impossible to deform a path in the Lie group to a point if the path in the Lie algebra cuts the sphere surface once, or an odd number of times. Yet another way to see the $2:1$ nature of the relation between the groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ is to follow the exponential of a diagonal operator in the Lie algebras' canonical form

$$\begin{aligned} \mathrm{Exp}\left(\tfrac{\mathrm{i}}{2}\begin{bmatrix} +\theta & 0 \\ 0 & -\theta \end{bmatrix}\right) &= \begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta/2} \end{bmatrix} \\ \mathrm{Exp}\left(\begin{bmatrix} +\mathrm{i}\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\theta \end{bmatrix}\right) &= \begin{bmatrix} \mathrm{e}^{+\mathrm{i}\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix}. \end{aligned} \tag{25}$$

The relation between the diagonal matrices presents the arguments above in their most elementary form. It is an approach we will use below to describe the homomorphisms among the other classical Lie groups.

These results are different aspects of a beautiful theorem by Cartan. Lie groups with isomorphic Lie algebras are locally isomorphic. There is a $1:1$ correspondence between Lie algebras and *simply connected* Lie groups. Every other Lie group with an isomorphic Lie algebra is isomorphic to the quotient of the simply connected Lie group by one of its discrete invariant subgroups.

Since the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are isomorphic, the groups are locally isomorphic. Since $\mathrm{SU}(2)$ is simply connected it *covers* all Lie groups with this Lie algebra. The maximal discrete invariant subgroup of $\mathrm{SU}(2)$ consists of all unimodular matrices that commute with $\mathrm{SU}(2)$ by Schur's Lemma: $c\mathrm{I}_2$. The unimodular condition requires $c^2 = +1$, so $c = \pm 1$. By Cartan's theorem, $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\mathrm{I}_2, -\mathrm{I}_2\}$ and therefore $\mathrm{SO}(3)$ is doubly connected.

The fundamental representation of $\mathfrak{su}(2)$ (A_1) has highest weight $\frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2)$ and is two-dimensional (cf Table 7). All representations of $\mathrm{SU}(2)$ can be constructed by exponentiating tensor representations of $\mathfrak{su}(2)$ with highest weight $(j, -j)$, $j = 0, 1/2, 1, 3/2, \cdots$. The (Wigner) representations $\mathcal{D}^j\,[\mathrm{SU}(2)]$ are faithful for j half-integer and $2:1$ representations if $j = l =$ integer. On the other hand the fundamental representation of $\mathfrak{so}(3)$ (B_1) has highest weight $\frac{1}{2}\mathbf{e}_1$ and is also two-dimensional. It exponentiates to matrices that are *not* representations of $\mathrm{SO}(3)$. Rather, they are faithful representations of the covering group $\widetilde{\mathrm{SO}(3)} = \mathrm{SU}(2)$ of $\mathrm{SO}(3)$. Only *half* of the tensor representations of this fundamental representation of $\mathrm{SO}(3)$, i.e., those representations $\mathcal{D}^j\,[\mathrm{SU}(2)]$ with $j = l =$ integer, are faithful representations of $\mathrm{SO}(3)$.

Similar relations hold among the other low-dimensional Lie groups and their algebras. For the compact real forms of the root spaces B_2 and C_2, the vectors in the root spaces, and those (*weights*) in the two fundamental representations, are

$$\begin{array}{cccc} & & \multicolumn{2}{c}{\text{Fundamental Reps.}} \\ & \text{Root Space} & \text{First} & \text{Second} \\ \hline B_2 & \pm\mathbf{e}_1 \pm \mathbf{e}_2, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \mathbf{0}, \mathbf{0} & \pm\mathbf{e}_1, \pm\mathbf{e}_2, \mathbf{0} & \frac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2) \\ C_2 & \pm\mathbf{e}_1 \pm \mathbf{e}_2, \pm 2\mathbf{e}_1, \pm 2\mathbf{e}_2, \mathbf{0}, \mathbf{0} & \pm\mathbf{e}_1, \pm\mathbf{e}_2 & \pm\mathbf{e}_1 \pm \mathbf{e}_2, \mathbf{0}. \end{array} \qquad (26)$$

The first fundamental representation of $B_2 = \mathrm{SO}(5)$ is five dimensional and the second is four-dimensional. Exponentiating diagonal operators in these two fundamental representations leads to 5×5 and 4×4 matrices

$$\begin{array}{l} \mathrm{Exp}(\phi_1 H_1 + \phi_2 H_2) \xrightarrow{1^{\mathrm{st}}} \begin{bmatrix} e^{i\phi_1} & & & & \\ 0 & e^{-i\phi_1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi_2} & 0 \\ 0 & 0 & 0 & 0 & e^{-i\phi_2} \end{bmatrix} \\ \mathrm{Exp}(\phi_1 H_1 + \phi_2 H_2) \xrightarrow{2^{\mathrm{nd}}} \begin{bmatrix} e^{\frac{i}{2}(+\phi_1+\phi_2)} & & & \\ 0 & e^{\frac{i}{2}(+\phi_1-\phi_2)} & 0 & 0 \\ 0 & 0 & e^{\frac{i}{2}(-\phi_1+\phi_2)} & 0 \\ 0 & 0 & 0 & e^{\frac{i}{2}(-\phi_1-\phi_2)} \end{bmatrix}. \end{array} \qquad (27)$$

If $\phi_1 \to \phi_1 + 2\pi$ the first fundamental representation maps to itself while the second maps to its negative. Similarly for the increase $\phi_2 \to \phi_2 + 2\pi$. The first fundamental representation is a faithful representation of $\mathrm{SO}(5)$ while the second fundamental representation is a faithful representation of its covering group $\mathrm{USp}(4, \mathbb{C}) = \widetilde{\mathrm{SO}(5)}$. The covering group is a double cover of $\mathrm{SO}(5)$.

For the first fundamental representation, points on the boundary $(\pm\pi, \phi_2) \cup (\phi_1, \pm\pi)$ map to identical points in the group and distinct interior points map to different operations in the group generated by these two commuting operators. As a result the square with edge length 2π (cf Fig. 1) parameterizes this abelian group. By contrast, points in the interior of the region $|\pm\phi_1 \pm \phi_2| < 2\pi$ map to distinct operations in the second fundamental representation of B_2 (cf Fig. 2), and points on the boundary map to $-\mathrm{I}_4$, in complete analogy with the $j = 1$ and $j = 1/2$ representations of $\mathrm{SU}(2)$. That the second fundamental representation of B_2 is a

$2:1$ cover of the first is clear because the area of the diamond in Fig. 2 is twice the area of the square in Fig. 1.

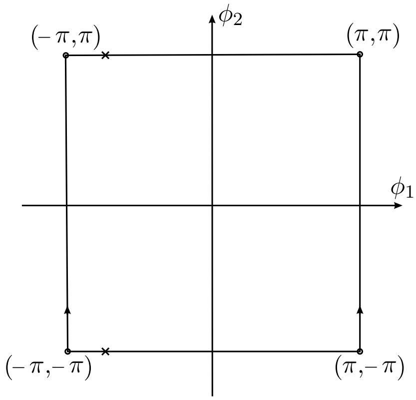

Figure 1. The interior of the square of edge length 2π maps onto the abelian subgroup of the first fundamental representation $\mathbf{F}_1$ of $\mathrm{SO}(5)$. The two points identified by a triangle map to the same group operation, as do the two points labeled by a cross.

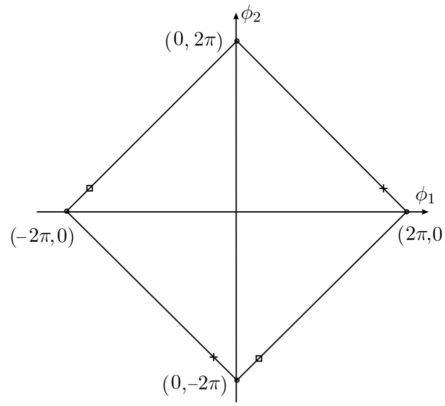

Figure 2. A square of twice the area maps onto the abelian subgroup of the second fundamental representation $\mathbf{F}_2$ of $\mathrm{SO}(5)$. The two points identified by a square map to the same group operation, as do the two points labeled by a plus sign. $\mathbf{F}_1$ is a 2 to 1 image of $\mathbf{F}_2$.

Identical relations hold for the two representations of C_2. The first fundamental representation is four dimensional and exponentiates to a faithful representation of $\mathrm{USp}(4,\mathbb{C})$. The second is five dimensional. Its exponential is a $2:1$ homomorphic image of $\mathrm{USp}(4,\mathbb{C})$ that is isomorphic to $\mathrm{SO}(5)$. The $2:1$ relation between the first and second fundamental representations of these two groups/algebras can also be determined directly from the weights of the two fundamental representations. Two basis weights for the first fundamental representation are $\mathbf{e}_1$ and $\mathbf{e}_2$. The other two weights are linear combinations of these roots with integer coefficients. The area enclosed by these weights is $\mathbf{e_1} \wedge \mathbf{e_2}$. By contrast, two basis weights in the second fundamental representation are $+\mathbf{e_1}+\mathbf{e_2}$ and $+\mathbf{e_1}-\mathbf{e_2}$, and the area enclosed by these weights is $(\mathbf{e_1}+\mathbf{e_2}) \wedge (\mathbf{e_1}-\mathbf{e_2}) = \mathbf{2e_1} \wedge \mathbf{e_2}$. The ratio of these areas is $1/2$. Since the phase angle parameter space ϕ is the dual, or reciprocal, space to that of the operators, the ratio of areas in the space of $\phi_1 - \phi_2$ cordinates is $2/1$. That is, the first representation is a two to one cover of the second. The first representation is also simply connected and faithful.

10. A_3 and D_3

Similar arguments apply to the three fundamental representations of D_3 and A_3. The two rank-three root spaces A_3 and D_3 are equivalent. This means that the complex extension Lie algebras associated with these two root spaces are isomorphic. There is a 1:1 correspondence among the five real forms (Table 6) of these Lie algebras. Since the root spaces are rank three, each real form Lie algebra has three fundamental irreducible representations. These representations are of dimension 4, 6, 4. They are unitary only for the compact real forms of these Lie algebras: $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$. We will compute the three fundamental irreducible representations for the root spaces (complex extension Lie algebras) A_3 and D_3 and then apply reality restrictions to determine the fundamental unitary irreducible representations of the compact real form algebras.

10.1. Equivalence Mapping

We begin by constructing a mapping between the two root spaces. The root space for the rank-three Lie algebra A_3 is most conveniently described in a three-dimensional subspace of a four-dimensional space with orthogonal basis vectors $\mathbf{e}_i,\ i = 1,2,3,4$. The nonzero roots in the root space are $\mathbf{e}_i - \mathbf{e}_j$, $1 \leq i \neq j \leq 4$. All nonzero roots are orthogonal to the vector $\mathbf{R} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$.

The nonzero roots for the rank-three Lie algebra D_3 have the form $\pm\mathbf{f}_i \pm \mathbf{f}_j$, $1 \leq i \neq j \leq 3$. A simple way to construct an equivalence between these two root spaces is to construct a transformation that maps $\mathbf{R}$ into a vector proportional to $\mathbf{f}_4$, which is orthogonal to all nonzero roots in D_3. Such a transformation is given by the rotation matrix

$$\mathcal{R} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{28}$$

Under this transformation we find

$$\begin{array}{ll} \mathbf{e}_1 - \mathbf{e}_2 \to \mathbf{f}_1 - \mathbf{f}_2, & \mathbf{e}_4 - \mathbf{e}_3 \to \mathbf{f}_1 + \mathbf{f}_2 \\ \mathbf{e}_1 - \mathbf{e}_3 \to \mathbf{f}_1 - \mathbf{f}_3, & \mathbf{e}_4 - \mathbf{e}_2 \to \mathbf{f}_1 + \mathbf{f}_3 \\ \mathbf{e}_2 - \mathbf{e}_3 \to \mathbf{f}_2 - \mathbf{f}_3, & \mathbf{e}_4 - \mathbf{e}_1 \to \mathbf{f}_2 + \mathbf{f}_3. \end{array} \tag{29}$$

10.2. Nonzero Matrix Elements

If we define the nonzero weights in an irreducible representation as $|\boldsymbol{\beta}\rangle$ and the shift operators by $E_{\boldsymbol{\alpha}}$, then

$$\langle \boldsymbol{\beta}'|E_{\boldsymbol{\alpha}}|\boldsymbol{\beta}\rangle = 0 \text{ if } \boldsymbol{\beta}' \neq \boldsymbol{\alpha}+\boldsymbol{\beta} \quad \text{and} \quad \langle \boldsymbol{\beta}'|E_{\boldsymbol{\alpha}}|\boldsymbol{\beta}\rangle \neq 0 \text{ if } \boldsymbol{\beta}' = \boldsymbol{\alpha}+\boldsymbol{\beta}. \tag{30}$$

The values of the matrix elements depend on the number of nonzero roots that have the form $\boldsymbol{\beta}' = \boldsymbol{\beta} + n\boldsymbol{\alpha}$. For the fundamental representations that we work with below the integer $n = \pm 1, 0$ and all nonzero matrix elements of the shift operators are equal.

10.3. Fundamental Representations of A_3

The Lie algebra A_3 has three fundamental irreducible representations. Two are four-dimensional and one is six-dimensional. For the compact real form $\mathfrak{su}(4)$, the four-dimensional representations are complex conjugates of each other and the six-dimensional representation is equivalent to a real representation.

From Table 7 the highest root of the first fundamental representation $\mathbf{F}_1$ is $\frac{1}{4}(3,2,1)$ $= \frac{3}{4}(\mathbf{e}_1 - \mathbf{e}_2) + \frac{2}{4}(\mathbf{e}_2 - \mathbf{e}_3) + \frac{1}{4}(\mathbf{e}_3 - \mathbf{e}_4) = \mathbf{e}_1 - \frac{1}{4}\mathbf{R}$. The four weights of this fundamental representation are $|i\rangle = \mathbf{e}_i - \frac{1}{4}\mathbf{R}$. By similar arguments the four weights in the other four-dimensional fundamental irreducible representation $\mathbf{F}_3$ are $\frac{1}{4}\mathbf{R} - \mathbf{e}_4$, $\frac{1}{4}\mathbf{R} - \mathbf{e}_3$, $\frac{1}{4}\mathbf{R} - \mathbf{e}_2$, $\frac{1}{4}\mathbf{R} - \mathbf{e}_1$.

The images of the operator $h_iH_i + A_{ij}E_{\mathbf{e}_i - \mathbf{e}_j}$ in these two representations are

$$\begin{aligned}
h_iH_i + A_{ij}E_{\mathbf{e}_i-\mathbf{e}_j} &\xrightarrow{\mathbf{F}_1} \begin{bmatrix} h_1-\lambda & A_{12} & A_{13} & A_{14} \\ A_{21} & h_2-\lambda & A_{23} & A_{24} \\ A_{31} & A_{32} & h_3-\lambda & A_{34} \\ A_{41} & A_{42} & A_{43} & h_4-\lambda \end{bmatrix} \\
h_iH_i + A_{ij}E_{\mathbf{e}_i-\mathbf{e}_j} &\xrightarrow{\mathbf{F}_3} \begin{bmatrix} \lambda-h_4 & A_{34} & A_{24} & A_{14} \\ A_{43} & \lambda-h_3 & A_{23} & A_{13} \\ A_{42} & A_{32} & \lambda-h_2 & A_{12} \\ A_{41} & A_{31} & A_{21} & \lambda-h_1 \end{bmatrix}.
\end{aligned} \tag{31}$$

For both representations $\lambda = \frac{1}{4}(h_1 + h_2 + h_3 + h_4)$. The diagonal elements indicate the basis vectors for the corresponding rows and columns. For example, in the fundamental representation $\mathbf{F}_1$, $h_1 - \lambda$ indicates that the weight for the first row/column is $\mathbf{e}_1 - \frac{1}{4}\mathbf{R}$.

For the six-dimensional fundamental irreducible representation the highest weight is $\frac{1}{4}(2,4,2) = \frac{1}{2}(+\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4)$. The $6 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ weights in this representation each have two positive and two negative coefficients. In this representation the image of the general operator in the Lie algebra is

$$h_i H_i + A_{ij} E_{\mathbf{e}_i - \mathbf{e}_j} \xrightarrow{\mathbf{F}_2} \begin{bmatrix} \frac{1}{2}D_{11} & A_{23} & A_{24} & A_{13} & A_{14} & 0 \\ A_{32} & \frac{1}{2}D_{22} & A_{34} & A_{12} & 0 & A_{14} \\ A_{42} & A_{43} & \frac{1}{2}D_{33} & 0 & A_{12} & A_{13} \\ A_{31} & A_{21} & 0 & \frac{1}{2}D_{44} & A_{34} & A_{24} \\ A_{41} & 0 & A_{21} & A_{43} & \frac{1}{2}D_{55} & A_{23} \\ 0 & A_{41} & A_{31} & A_{42} & A_{32} & \frac{1}{2}D_{66} \end{bmatrix}. \qquad (32)$$

Once again, the diagonal matrix elements $D_{ii}, I = 1, ..., 6$, are

$$\begin{aligned} D_{11} &= \frac{1}{2}(h_1 + h_2 - h_3 - h_4), & D_{44} &= \frac{1}{2}(-h_1 + h_2 + h_3 - h_4) \\ D_{22} &= \frac{1}{2}(h_1 - h_2 + h_3 - h_4), & D_{55} &= \frac{1}{2}(-h_1 + h_2 - h_3 + h_4) \\ D_{33} &= \frac{1}{2}(h_1 - h_2 - h_3 + h_4), & D_{66} &= \frac{1}{2}(-h_1 - h_2 + h_3 + h_4). \end{aligned}$$

10.4. Fundamental Representations of D_3

From Table 7 the highest weight of the first fundamental irreducible representation $\mathbf{F}_1$ is $\mathbf{f}_1$. The set of six weights for this representation consists of $\pm\mathbf{f}_i$, $i = 1, 2, 3$. The image of the operator $h_i H_i + B_{ij} E_{\pm\mathbf{f}_i \pm \mathbf{f}_j}$ in this six-dimensional representation is

$$h_i H_i + B_{ij} E_{\pm\mathbf{f}_i \pm \mathbf{f}_j} \xrightarrow{\mathbf{F}_1} \begin{bmatrix} h_1 & B_{(+-0)} & B_{(+0-)} & B_{(+0+)} & B_{(++0)} & 0 \\ B_{(-+0)} & h_2 & B_{(0+-)} & B_{(0++)} & 0 & B_{(++0)} \\ B_{(-0+)} & B_{(0-+)} & h_3 & 0 & B_{(0++)} & B_{(+0+)} \\ B_{(-0-)} & B_{(0--)} & 0 & -h_3 & B_{(0+-)} & B_{(+0-)} \\ B_{(--0)} & 0 & B_{(0--)} & B_{0-+)} & -h_2 & B_{(+-0)} \\ 0 & B_{(--0)} & B_{(-0-)} & B_{(-0+)} & B_{(-+0)} & -h_1 \end{bmatrix}. \qquad (33)$$

In this matrix $B_{(+0+)}$ is the coefficient of $E_{+\mathbf{f}_1+\mathbf{f}_3}$. As usual, the diagonal matrix elements are markers for the weights in this representation.

The highest weight of the second fundamental irreducible representation $\mathbf{F}_2$ is $\frac{1}{2}(\mathbf{f}_1 + \mathbf{f}_2 - \mathbf{f}_3)$. The four weights in this representation are $\frac{1}{2}(\pm\mathbf{f}_1 \pm \mathbf{f}_2 \pm \mathbf{f}_3)$, with an odd number of minus signs. For the third fundamental irreducible representation $\mathbf{F}_3$ the set of four weights is similar, but with an even number of minus signs [21].

The image of the operator $h_iH_i + B_{ij}E_{\pm\mathbf{f}_i\pm\mathbf{f}_j}$ in the four-dimensional representation $\mathbf{F}_2$ is

$$h_iH_i + B_{ij}E_{\pm\mathbf{f}_i\pm\mathbf{f}_j} \xrightarrow{\mathbf{F}_2} \begin{bmatrix} B^2_{11} & B_{(0+-)} & B_{(+0-)} & B_{(++0)} \\ B_{(0-+)} & B^2_{22} & B_{(+-0)} & B_{(+0+)} \\ B_{(-0+)} & B_{(-+0)} & B^2_{33} & B_{(0++)} \\ B_{(--0)} & B_{(-0-)} & B_{(0--)} & B^2_{44} \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} B^2_{11} &= \frac{1}{2}(h_1 + h_2 + h_3), & B^2_{22} &= \frac{1}{2}(h_1 - h_2 + h_3) \\ B^2_{33} &= \frac{1}{2}(-h_1 + h_2 + h_3), & B^2_{44} &= \frac{1}{2}(-h_1 - h_2 - h_3) \end{aligned}$$

while in the other four-dimensional representation $\mathbf{F}_3$ it is

$$h_iH_i + B_{ij}E_{\pm\mathbf{f}_i\pm\mathbf{f}_j} \xrightarrow{\mathbf{F}_3} \begin{bmatrix} B^3_{11} & B_{(0++)} & B_{(+0+)} & B_{(++0)} \\ B_{(0--)} & B^3_{22} & B_{(+-0)} & B_{(+0-)} \\ B_{(-0-)} & B_{(-+0)} & B^3_{33} & B_{(0+-)} \\ B_{(--0)} & B_{(-0+)} & B_{(0-+)} & B^3_{44} \end{bmatrix} \tag{35}$$

with

$$\begin{aligned} B^3_{11} &= \frac{1}{2}(h_1 + h_2 + h_3), & B^3_{22} &= \frac{1}{2}(h_1 - h_2 + h_3) \\ B^3_{33} &= \frac{1}{2}(-h_1 + h_2 + h_3), & B^3_{44} &= \frac{1}{2}(-h_1 - h_2 - h_3). \end{aligned}$$

10.5. Unitary Representations of Compact Groups

In order for the matrices in equations (31-35) to exponentiate to unitary representations of the compact groups $\mathrm{SU}(4)$ and $\mathrm{SO}(6)$ the matrices must be antihermitian. This condition places constraints on the complex coefficients: $h_i^* = -h_i$ and $A^*_{ij} = -A_{ji}$ for $\mathfrak{su}(4)$ and similarly for the coefficients in the Lie algebra $\mathfrak{so}(6)$. Using these constraints it is possible to show that the two four-dimensional fundamental irreducible representations of $\mathfrak{su}(4)$ are related to each other by complex conjugation and the six-dimensional representation is equivalent to a real antisymmetric representation of the Lie algebra. This is true also for the two four-dimensional 'spinor' representations of $\mathfrak{so}(6)$. The six-dimensional representation is equivalent to the Lie algebra of real antisymmetric matrices obtained by linearization of the Lie group $\mathrm{SO}(6)$ about the identity, which consists of real antisymmetric infinitesimal matrices. For all representations of these Lie algebras the trace is zero: this condition exponentiates to the condition that the determinant of the corresoponding group operation is $+1$.

11. Computational Simplifications

Relations among the low-dimensional Lie groups and their algebras can sometimes be used to simplify computations. The general idea is to construct a global parameterization of the Lie group by exponentiating points in the Lie algebra, and then use matrix multiplication to construct an analytic expression for the composition of two group operations [26]. The group composition law is matrix multiplication, which is already very simple, so there is not much room for further simplification. For semisimple groups consisting of $n \times n$ matrices with $n > 2$ the results are generally uninstructive. However, when the Lie group is locally isomorphic to a 2×2 matrix Lie group some elegant results are possible. Two cases occur that are of particular interest in physical applications: $\mathrm{SU}(2) \downarrow \mathrm{SO}(3)$ and $\mathrm{SL}(2,\mathbb{C}) \downarrow \mathrm{SO}(3,1)$.

11.1. Rotations and $\mathrm{SU}(2)$

The homomorphism that exists between $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ has already been discussed Section 9. It is possible to describe the composition of two rotation group operations by mapping each (nonuniquely) to an $\mathrm{SU}(2)$ counterpart, constructing the product of these 2×2 matrices, and then mapping the resulting operation in $\mathrm{SU}(2)$ back to $\mathrm{SO}(3)$ (uniquely). To be specific, we map the rotations represented by $(\hat{\mathbf{n}}_1, \theta_1)$ and $(\hat{\mathbf{n}}_2, \theta_2)$ to their $\mathrm{SU}(2)$ counterparts and compute the product

$$\left(\cos\frac{\theta_1}{2}\mathrm{I}_2 + \mathrm{i}\sin\frac{\theta_1}{2}\hat{\mathbf{n}}_{\mathbf{1}}\cdot\boldsymbol{\sigma}\right)\left(\cos\frac{\theta_2}{2}\mathrm{I}_2 + \mathrm{i}\sin\frac{\theta_2}{2}\hat{\mathbf{n}}_{\mathbf{2}}\cdot\boldsymbol{\sigma}\right) = \cos\frac{\theta}{2}\mathrm{I}_2 + \mathrm{i}\sin\frac{\theta}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}. \tag{36}$$

Expanding the equation on the left yields simple expressions for the rotation angle θ and axis $\hat{\mathbf{n}}$

$$\cos\frac{\theta}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{\mathbf{n}}_{\mathbf{1}}\cdot\hat{\mathbf{n}}_{\mathbf{2}} \tag{37a}$$

$$\sin\frac{\theta}{2}\hat{\mathbf{n}} = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\hat{\mathbf{n}}_1 + \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\hat{\mathbf{n}}_2 - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\hat{\mathbf{n}}_1\times\hat{\mathbf{n}}_2. \tag{37b}$$

This result can be expressed in a somewhat more symmetric way as follows

$$\tan\frac{\theta}{2}\hat{\mathbf{n}} = \frac{\tan\frac{\theta_1}{2}\hat{\mathbf{n}}_1 + \tan\frac{\theta_2}{2}\hat{\mathbf{n}}_2 - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\hat{\mathbf{n}}_1\times\hat{\mathbf{n}}_2}{1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2} \tag{37c}$$

or in a yet more memorable way

$$\mathbf{c} = \frac{\mathbf{a}+\mathbf{b}-\mathbf{a}\times\mathbf{b}}{1-\mathbf{a}\cdot\mathbf{b}}, \quad \mathbf{c} = \tan\frac{\theta}{2}\hat{\mathbf{n}}, \quad \mathbf{a} = \tan\frac{\theta_1}{2}\hat{\mathbf{n}}_1, \quad \mathbf{b} = \tan\frac{\theta_2}{2}\hat{\mathbf{n}}_2. \tag{37d}$$

The information in equations (37) then defines the product of rotations in $\mathrm{SO}(3)$ – $(\hat{\mathbf{n}}_1, \theta_1)\cdot(\hat{\mathbf{n}}_2, \theta_2) = (\hat{\mathbf{n}}, \theta)$.

The rotation group $\mathrm{SO}(3)$ can be parameterized in many different ways. Each different parameterization leads to its own unique analytic expression for the group composition law. In principle all parameterizations are equivalent. These remarks are valid for all Lie groups. Among various parameterizations of any Lie group one is more equal than others. This is the exponential parameterization. This is the parameterization obtained by mapping the Lie algebra onto the Lie group using the exponential mapping. The analytic expression in equation (37) is obtained using this unique parameterization. Analytic reparameterizations of the group composition law due to various alternative parameterizations of the Lie group will be discussed in Section 12 for Lie groups in general, and Subsection 12.4 for $\mathrm{SO}(3)$ in particular.

11.2. Lorentz Transformations and $\mathrm{SL}(2,\mathbb{C})$

Lorentz transformations leave invariant the quadratic form $x^2+y^2+z^2-(ct)^2$. Lorentz transformations are elements in the Lie group $\mathrm{O}(3,1)$, consisting of four disconnected components. The component connected to the identity, I_4, is locally isomorphic to the group of 2×2 complex matrices $\mathrm{SL}(2,\mathbb{C})$. A general element in the Lie algebra $\mathfrak{so}(3,1)$ has the form

$$\theta_i L_i + w_j B_j = \left[\begin{array}{ccc|c} 0 & \theta_3 & -\theta_2 & -w_1 \\ -\theta_3 & 0 & \theta_1 & -w_2 \\ \theta_2 & -\theta_1 & 0 & -w_3 \\ \hline -w_1 & -w_2 & -w_3 & 0 \end{array}\right] \tag{38}$$

$$[L_i, L_j] = -\epsilon_{ijk}L_k, \qquad [L_i, B_j] = -\epsilon_{ijk}B_k, \qquad [B_i, B_j] = +\epsilon_{ijk}L_k.$$

The commutation relations of the generators L_i, B_j are given on the right in equation (39).

The interpretation of the boost operators can be obtained by computing the action of any one of them on the coordinates (x, y, z, ct)

$$\begin{aligned}\begin{bmatrix} x' \\ y' \\ z' \\ ct' \end{bmatrix} &= \mathrm{Exp}(w_3 B_3) \begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix} \\ &= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh w_3 & -\sinh w_3 \\ \hline 0 & 0 & -\sinh w_3 & \cosh w_3 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix} = \begin{bmatrix} x \\ y \\ \gamma(z-\beta ct) \\ \gamma(ct-\beta z) \end{bmatrix}.\end{aligned} \tag{39}$$

The boost $\mathrm{Exp}(w_3 B_3)$ transforms from one frame to another moving with parallel axes and with velocity $(0, 0, v)$ with respect to the first frame. In these expressions $\gamma = \cosh w_3, \gamma\beta = \sinh w_3, \beta = v/c, \gamma = 1/\sqrt{1-\beta^2}$.

The relation between the groups $\mathrm{SO}(3, 1)$ and $\mathrm{SL}(2, \mathbb{C})$ is obtained using the commutation relations of the Pauli spin matrices $[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k$. From this we see the correspondence

$$\mathbf{L} \leftrightarrow \frac{\mathrm{i}}{2}\boldsymbol{\sigma}, \qquad \mathbf{B} \leftrightarrow \frac{1}{2}\boldsymbol{\sigma}. \tag{40}$$

In this representation the boost is given by

$$\mathrm{Exp}(\mathbf{w} \cdot \mathbf{B}) = \mathrm{Exp}\left(\frac{1}{2} w \hat{\mathbf{w}} \cdot \boldsymbol{\sigma}\right) = \cosh\frac{w}{2}\mathrm{I}_2 + \sinh\frac{w}{2}\hat{\mathbf{w}}\cdot\boldsymbol{\sigma} \tag{41}$$

where $\mathbf{w} = w\hat{\mathbf{w}}$. A general element in $\mathrm{SO}(3, 1)$ can be written as an element in the coset $\mathrm{SO}(3, 1)/\mathrm{SO}(3)$, which is a boost, together with a rotation

$$\begin{aligned}\mathrm{SO}(3,1) &= \mathrm{Exp}(\mathbf{w} \cdot \mathbf{B})\mathrm{Exp}(\boldsymbol{\theta}\cdot\mathbf{L}) \to \mathrm{SL}(2,\mathbb{C}) \\ &= \left(\cosh\frac{w}{2}\mathrm{I}_2 + \sinh\frac{w}{2}\hat{\mathbf{w}}\cdot\boldsymbol{\sigma}\right)\left(\cos\frac{\theta}{2}\mathrm{I}_2 + \mathrm{i}\sin\frac{\theta}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\right).\end{aligned} \tag{42}$$

Any 2×2 matrix $m \in \mathrm{SL}(2, \mathbb{C})$ can be written in the form defined by this coset decomposition. One way is to multiply out the product given in equation (43) into the form $M = \sum_{\mu=0}^{3} M_\mu \sigma_u$ and compare these complex components M_μ with those of m. Another procedure is to construct the boost directly through

$$\begin{aligned}mm^\dagger &= \mathrm{Exp}(\mathbf{w} \cdot \mathbf{B})\mathrm{Exp}(\boldsymbol{\theta}\cdot\mathbf{L})\mathrm{Exp}(\boldsymbol{\theta}\cdot\mathbf{L})^\dagger\mathrm{Exp}(\mathbf{w} \cdot \mathbf{B})^\dagger \\ &= (\mathrm{Exp}(\mathbf{w} \cdot \mathbf{B}))^2 = \mathrm{Exp}(2\mathbf{w} \cdot \mathbf{B}).\end{aligned} \tag{43}$$

The value of $2\mathbf{w}$, and therefore $\mathbf{w}$, can be determined directly from equation (41).

In order to illustrate the convenience of this procedure we compute the product of two boosts $(w_1, \hat{\mathbf{w}}_1) \cdot (w_2, \hat{\mathbf{w}}_2) = (w, \hat{\mathbf{w}}) \cdot (\theta, \hat{\mathbf{n}})$. This is done by multiplying out the representatives of the boosts in $\mathrm{SL}(2, \mathbb{C})$

$$\begin{aligned}&\mathrm{Exp}(\mathbf{w_1} \cdot \mathbf{B})\mathrm{Exp}(\mathbf{w_2} \cdot \mathbf{B}) \\ &\quad = \left(\cosh \frac{w_1}{2} \mathrm{I}_2 + \sinh \frac{w_1}{2} \hat{\mathbf{w}}_\mathbf{1} \cdot \boldsymbol{\sigma}\right) \left(\cosh \frac{w_2}{2} \mathrm{I}_2 + \sinh \frac{w_2}{2} \hat{\mathbf{w}}_\mathbf{2} \cdot \boldsymbol{\sigma}\right)\end{aligned} \tag{44}$$

and comparing with the terms in equation (43). We find

$$\begin{aligned}\hat{\mathbf{w}} \cdot \hat{\mathbf{n}} &= \mathbf{0}, & \mathbf{a} &= \tanh \frac{w_1}{2} \hat{\mathbf{w}}_1 \\ \tan \frac{\theta}{2} \hat{\mathbf{n}} &= \frac{\mathbf{a} \times \mathbf{b}}{1 + \mathbf{a} \cdot \mathbf{b}}, & \mathbf{b} &= \tanh \frac{w_2}{2} \hat{\mathbf{w}}_2 \\ \cosh w &= (1 + \mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) \\ \hat{\mathbf{w}} &\simeq \mathbf{a} + \mathbf{b} + \frac{(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})}{2(1 + \mathbf{a} \cdot \mathbf{b})}\cdot\end{aligned} \tag{45}$$

In the nonrelativistic limit $\mathbf{w} = w\hat{\mathbf{w}} = \mathbf{v}/c$ and we find to first order

$$\boldsymbol{\theta} = \frac{1}{2} \frac{\mathbf{v}_1}{c} \times \frac{\mathbf{v}_2}{c}, \qquad \frac{\mathbf{v}}{c} = \frac{\mathbf{v}_1}{c} + \frac{\mathbf{v}_2}{c}\cdot \tag{46}$$

In particular, if one of the boosts is infinitesimal ($\boldsymbol{\beta}_2 \to \delta\boldsymbol{\beta}_2$), then the rotation angle is also infinitesimal and is

$$\delta\Theta = \frac{1}{2} \boldsymbol{\beta}_1 \times \delta\boldsymbol{\beta}_2\ . \tag{47}$$

This angular precession is called the Thomas precession [54].

11.3. $\mathrm{SO}(3, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$

The compact simple Lie group $\mathrm{SO}(3, \mathbb{R})$ has a complex extension $SO(3; \mathbb{C})$ in which each of the real parameters in $\mathrm{SO}(3, \mathbb{R})$ is replaced by a complex number. At the level of the Lie algebra of 3×3 matrices, the three compact generators are the 3×3 angular momentum matrices $\mathbf{L}$ defined in equation (21). The three noncompact generators can be chosen as $\mathbf{B} = \mathrm{i}\mathbf{L}$ ($\mathbf{B}$ are not to be confused with the operators $\mathbf{B}$ defined in equation (39)). The commutation relations are

$$
\begin{aligned}
[L_i, L_j] &= -\epsilon_{ijk} L_k, & [s_i, s_j] &= -\epsilon_{ijk} s_k \\
[L_i, B_j] &= -\epsilon_{ijk} B_k, & [s_i, b_j] &= -\epsilon_{ijk} b_k \\
[B_i, B_j] &= +\epsilon_{ijk} L_k, & [b_i, b_j] &= +\epsilon_{ijk} s_k.
\end{aligned} \tag{48}
$$

The globally homomorphic and locally isomorphic Lie group $\mathrm{SU}(2)$ also has a complex extension, obtained the same way. The six infinitesimal generators for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ can be taken as $\mathbf{s}_j = \frac{\mathrm{i}}{2}\boldsymbol{\sigma}_j$ and $\mathbf{b}_j = \mathrm{i}\,\frac{\mathrm{i}}{2}\boldsymbol{\sigma}_j$. The commutation relations among these six infinitesimal generators are given above in equation (48). The commutation relations for these two Lie algebras are those of the homogeneous Lorentz group $\mathrm{SO}(3,1)$ given in equation (39).

There are two ways to create isomorphisms between $\mathfrak{so}(3;\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$. These are

$$
\begin{array}{lll}
I & \mathbf{L} \leftrightarrow \mathbf{s}, & \mathbf{B} \leftrightarrow +\mathbf{b} \\
II & \mathbf{L} \leftrightarrow \mathbf{s}, & \mathbf{B} \leftrightarrow -\mathbf{b}\,.
\end{array} \tag{49}
$$

The Exponential map can be used to construct a mapping from the Lie algebras back to the Lie groups. For $\mathrm{SO}(3,\mathbb{C})$ we find

$$
\begin{aligned}
\mathrm{Exp}(\boldsymbol{\theta}\cdot\mathbf{L} + \boldsymbol{\gamma}\cdot\mathbf{B}) &= \mathrm{Exp}\left((\theta + \mathrm{i}\gamma)_i L_i\right) \\
&= \mathrm{I}_3 + \sin(\theta + \mathrm{i}\gamma)\cdot\mathbf{L}(\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma}) + (1 - \cos(\theta + \mathrm{i}\gamma))\left((\boldsymbol{\theta} + \mathrm{i}\gamma)\cdot\mathbf{L}\right)^2 .
\end{aligned} \tag{50}
$$

This result is a simple extension of equation (24). It results from exploiting the Cayley-Hamilton theorem: every matrix satisfies its secular equation.

The exponentiation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ to the Lie group $\mathrm{SL}(2,\mathbb{C})$ follows the same procedure

$$
\begin{aligned}
\mathrm{Exp}(\boldsymbol{\theta}\cdot\mathbf{s} + \boldsymbol{\gamma}\cdot\mathbf{b}) &= \mathrm{Exp}\left(\mathrm{i}(\theta + \mathrm{i}\gamma)_j \sigma_j/2\right) \\
&= \cos\frac{\theta + \mathrm{i}\gamma}{2}\mathrm{I}_2 + \mathrm{i}\sin\frac{\theta + \mathrm{i}\gamma}{2}\cdot(\widehat{\boldsymbol{\theta} + \mathrm{i}\gamma}).
\end{aligned} \tag{51}
$$

In the event that $(\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma})\cdot(\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma}) = 0$ the expansion equation (51) for $\mathrm{SO}(3,\mathbb{C})$ terminates at the third (quadratic) term and that for $\mathrm{SL}(2,\mathbb{C})$ terminates at the second (linear) term.

The topological properties of these Lie groups are easily determined. For semisimple Lie groups $\mathfrak{g}$ with maximal compact subalgebra $\mathfrak{k}$ and Cartan decomposition

$$
\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \qquad [\mathfrak{k},\mathfrak{k}] = \mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{p}] = \mathfrak{p}, \qquad [\mathfrak{p},\mathfrak{p}] = (-)\,\mathfrak{k} \tag{52}
$$

the exponential from the algebra to the group can be parameterized by a coset decomposition: $\mathrm{Exp}(\mathfrak{g}) \to G = K * P$. The space $P = \mathrm{Exp}(\mathfrak{p})$ is simply connected and topologically equivalent to $\mathbb{R}^n$ for some appropriate $n = \dim(\mathfrak{p})$, $n = 3$ in the current cases. The connectivity of G is therefore the same as the connectivity of the maximal compact subgroup $K = \mathrm{Exp}(\mathfrak{k})$. In the present cases the maximal compact subgroups are $\mathrm{SO}(3, \mathbb{R}) \subset \mathrm{SO}(3, \mathbb{C})$, which is doubly connected, and $\mathrm{SU}(2)) \subset \mathrm{SL}(2, \mathbb{C})$, which is simply connected.

The composition law for operations in these two groups can be computed in the exponential parameterization. Since the two groups are homomorphic it is sufficient to construct the group composition law in the double cover group: $\mathrm{SL}(2, \mathbb{C})$. This is done by composing 2×2 matrices of the form given in equation (52). For all practical purposes this has already been done in equations (36) and (37). The following substitutions effect this group composition law

$$\boldsymbol{\theta} = (\hat{\mathbf{n}}, \theta) \to \boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma} = \left(\frac{\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma}}{|\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma}|}, |\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma}| \right)$$

where

$$|\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma}| = \sqrt{(\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma}) \cdot (\boldsymbol{\theta} + \mathrm{i}\boldsymbol{\gamma})}. \tag{53}$$

The vectors $\mathbf{a}, \mathbf{b}$ parameterizing the two operations in $\mathrm{SU}(2)$ are to be replaced by the corresponding complex three-vectors parameterizing the two operations in the group $\mathrm{SL}(2, \mathbb{C})$, and their product is given by the complex three-vector corresponding to $\mathbf{c}$ in equation (37).

11.4. $D_2 = A_1 \oplus A_1$

The Lie algebras with rank-two root space D_2 have two diagonal operators H_1, H_2 and four nonzero roots $\pm \mathbf{e}_1 \pm \mathbf{e}_2$. The root space is semisimple: it describes two mutually commuting rank one Lie algebras of type $A_1 = B_1 = C_1$ with three operators each: $H_1 + H_2$, $E_{\pm(\mathbf{e}_1 + \mathbf{e}_2)}$ and $H_1 - H_2$, $E_{\pm(\mathbf{e}_1 - \mathbf{e}_2)}$. If we describe these two three-dimensional Lie algebras as $\mathbf{A}$ and $\mathbf{B}$ then the commutation relations can be summarized as follows

$$\mathbf{A} \times \mathbf{A} = -\mathbf{A}, \qquad \mathbf{B} \times \mathbf{B} = -\mathbf{B}, \qquad \mathbf{A} \times \mathbf{B} = \mathbf{0}. \tag{54}$$

The two Lie algebras can be constructed from the six infinitesimal generators of the matrix group $\mathrm{SO}(4)$: these consist of antisymmetric 4×4 matrices that can be taken as

$$\boldsymbol{\theta}\cdot\mathbf{J} = \left[\begin{array}{ccc|c} 0 & \theta_3 & -\theta_2 & 0 \\ -\theta_3 & 0 & \theta_1 & 0 \\ \theta_2 & -\theta_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \qquad \boldsymbol{\gamma}\cdot\mathbf{K} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & \gamma_1 \\ 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & \gamma_3 \\ \hline -\gamma_1 & -\gamma_2 & -\gamma_3 & 0 \end{array}\right]. \tag{55}$$

The operators J_i, K_j satisfy the commutation relations

$$\begin{aligned} [J_i, J_j] &= -\,\epsilon_{ijk} J_k, & [J_i, K_j] &= -\,\epsilon_{ijk} K_k \\ [K_i, J_j] &= -\,\epsilon_{ijk} K_k, & [K_i, K_j] &= -\,\epsilon_{ijk} J_k \end{aligned} \tag{56}$$

which can be written in simplified vector notation as

$$\mathbf{J}\times\mathbf{J} = -\mathbf{J}, \quad \mathbf{J}\times\mathbf{K} = -\mathbf{K}, \quad \mathbf{K}\times\mathbf{J} = -\mathbf{K}, \quad \mathbf{K}\times\mathbf{K} = -\mathbf{J}. \tag{57}$$

The two three-dimensional Lie algebras $\mathbf{A}, \mathbf{B}$ can be expressed in terms of the matrices J_i, K_j as follows

$$\begin{gathered} \mathbf{A} = \tfrac{1}{2}(\mathbf{J}+\mathbf{K}), \qquad \mathbf{B} = \tfrac{1}{2}(\mathbf{J}-\mathbf{K}) \\ \mathbf{A}\times\mathbf{A} = -\mathbf{A}, \qquad \mathbf{B}\times\mathbf{B} = -\mathbf{B}, \qquad \mathbf{A}\times\mathbf{B} = \mathbf{0}. \end{gathered} \tag{58}$$

11.5. Laboratory and Body System Rotations: $\mathrm{SO}(4)$

The rotational properties of molecules can be described by rotation operators $\mathbf{J}$ in either the laboratory (inertial) coordinate system or rotational operators $\mathbf{L}$ in a body-fixed (noninertial) coordinate system. The two sets of operators are related by a rotation matrix

$$L_\alpha = \langle \mathbf{e}_\alpha | \mathbf{e}_i \rangle J_i = \langle \alpha | i \rangle J_i\ . \tag{59}$$

Here $\mathbf{e}_i$ are orthonormal basis vectors in a laboratory fixed coordinate system, $\mathbf{e}_\alpha$ are orthonormal basis vectors in a body-fixed frame, and $\langle \mathbf{e}_\alpha | \mathbf{e}_i \rangle = \langle \alpha | i \rangle$ are matrix elements (i.e., direction cosines) in an orthogonal transformation from one system to the other and $1 \leq i, j, k \leq 3$, $1 \leq \alpha, \beta, \gamma \leq 3$. Klein observed [40] that the commutation relations among the angular momentum operators J_i in the laboratory-fixed coordinate system were opposite those of the operators L_α in the body-fixed frame, that the two sets commuted with each other, and that the total angular momentum was the same in both coordinate systems. An extensive discussion of the properties of these operators was given in Casimir's thesis [10]. The relationship among these operators was later reviewed by van Vleck [55] from a slightly different viewpoint.

The relations among the operators $\mathbf{J}$ and $\mathbf{L}$ are ($\hbar = 1$)

$$\mathbf{L} \cdot \mathbf{L} = \sum_{\alpha} L_{\alpha}^2 = (\langle \alpha | i \rangle J_i)^t (\langle \alpha | j \rangle J_j) = J_i \delta_{ij} J_j = \sum_i J_i^2 = \mathbf{J} \cdot \mathbf{J} \tag{60}$$

and

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k, \qquad [L_\alpha, L_\beta] = -\mathrm{i}\epsilon_{\alpha\beta\gamma} L_\gamma, \qquad [J_i, L_\beta] = 0. \tag{61}$$

The commutation properties of the body-fixed operators with themselves and with the laboratory-fixed operators are obtained from the following relations among the direction cosines $\langle \alpha | i \rangle$ and the operators $\mathbf{J}$

$$[\langle \alpha | \mathrm{i} \rangle, J_j] = i\epsilon_{ijk} \langle \gamma | k \rangle, \qquad \epsilon_{ijk} \langle \alpha | i \rangle \langle \beta | j \rangle = \epsilon_{\alpha\beta\gamma} \langle \gamma | k \rangle. \tag{62}$$

Using these relations it is a straightforward if tedious exercise [55] to show that the commutation relations of $\mathbf{J}$ with $\mathbf{J}$ and of $\mathbf{L}$ with $\mathbf{L}$ have opposite signs and that $\mathbf{J}$ and $\mathbf{L}$ commute, as stated in equation (61). These operators together span a Lie algebra of type $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

The hermitian operators $\mathbf{J}$ and $\mathbf{L}$ can be related to the antisymmetric operators $\mathbf{A}$ and $\mathbf{B}$ in equation (58) as follows

$$\mathbf{J} = -\mathrm{i}\mathbf{A}, \qquad \mathbf{L} = -\mathrm{i}\mathbf{B}. \tag{63}$$

The two independent Lie algebras spanned by $\mathbf{J}$ and $\mathbf{L}$, or $\mathbf{A}$ and $\mathbf{B}$, have representations indexed by quantum numbers j_a, j_b, with $2j_a = 0, 1, 2, \cdots$ and $2j_b = 0, 1, 2, \cdots$ (incoherent). However, the requirement that $\mathbf{J} \cdot \mathbf{J} = \mathbf{L} \cdot \mathbf{L}$ restricts the class of representations to the subset $j_a = j_b$ in studies of the rotational properties of molecules.

12. Analytic Reparameterizations

The operations in a Lie group are identified with points in a manifold. There are as many ways to parameterize Lie group operations as there are ways to introduce coordinate systems on a manifold, viz: infinite. Some parameterizations are more useful than others. One among them is unique: the exponential parameterization maps points in the Lie algebra to elements in the Lie group in a locally 1:1 way and in a globally many to one way. The group composition and inversion laws are expressed in terms of analytic functions. Different parameterizations require different analytic functions. The construction of analytic functions that represent group composition and inversion laws can be a nightmare. As a particular example, if A and B are elements in the Lie algebra of some Lie group, then e^A and e^B are operations in the Lie group. Therefore $\mathrm{e}^A \cdot \mathrm{e}^B$ is a group operation that we can represent in the form e^C, for some possibly nonunique element $C = C(A, B)$ in the Lie algebra. The computation of C is the famous Baker-Campbell-Hausdorff problem [5,8,35]. A formal solution to this problem was given by Dynkin [14,15]. It is as elegant mathematically as it is useless computationally.

The purpose of this Section is to introduce a simple *trick* (i.e., theorem) that reduces this computation to a simple algorithm [27,29]. In short: 1) find a faithful matrix representation of the Lie algebra; 2) carry out the exponentiations using this representation; 3) carry out the group multiplications using these matrices; 4) determine the element in the matrix Lie algebra that gives this result; 5) Viola! This result is true in the abstract Lie group as well as in all its matrix representations.

This algorithm will be illustrated first by applications to the Heisenberg algebras/groups, which are not simple and have therefore not been discussed previously in this work. It will then be applied to $\mathrm{SU}(2)$ in two different ways. We close this Section with an application to $\mathrm{SU}(4)$.

12.1. Heisenberg Nilpotent Group

The Heisenberg group H_3 is a three-dimensional nilpotent Lie group. A basis in the Lie algebra $\mathfrak{h}_3$ that is very useful in the Quantum Theory consists of the creation and annihilation operators $a^\dagger = \frac{1}{\sqrt{2}}(x-D)$ (where $D = \mathrm{d}/\mathrm{d}x$) and $a = \frac{1}{\sqrt{2}}(x+D)$ and their commutator $\left[a, a^\dagger\right] = I$. These operators act in an infinite-dimensional Hilbert space whose basis vectors are traditionally chosen as the number states $|n\rangle$, $n = 0, 1, 2, \cdots$, with $a|n\rangle = |n-1\rangle\sqrt{n}$, $a^\dagger|n\rangle = |n+1\rangle\sqrt{n+1}$, and $I|n\rangle = 1|n\rangle$. The algebra $\mathfrak{h}_3$ acts on the Hilbert space through an infinite-dimensional representation.

The algebra has many other representations. One of them consists of 3×3 matrices [27,29], with

$$\Gamma^f(La + Ra^\dagger + DI) = \begin{bmatrix} 0 & L & D \\ 0 & 0 & R \\ 0 & 0 & 0 \end{bmatrix}. \tag{64}$$

The 3×3 matrix representatives of the basis operators satisfy the same commutation relations as the operators $(\left[\Gamma^f(a), \Gamma^f(a^\dagger)\right] = \Gamma^f(\left[a, a^\dagger\right]) = \Gamma^f(I))$ and provide a faithful matrix representation of this algebra.

For many purposes it is useful to be able to reparameterize a group operation of the form $\mathrm{e}^{\alpha a^\dagger + \beta a}$ in such a way that all the annihilation operators act first and all the creation operators act last

$$e^{\alpha a^\dagger+\beta a} \stackrel{?}{=} e^{\alpha' a^\dagger}\ e^{d' I}\ e^{\beta' a}. \tag{65}$$

The result on the right is an element in the Lie group, and requires the presence of the additional group operation $e^{d'I}$. The exponentials are easily carried out in the faithful three-dimensional representation

$$\mathrm{Exp}\left(\Gamma^f(La+Ra^\dagger+DI)\right)\to \mathrm{Exp}\left(\begin{bmatrix}0&L&D\\0&0&R\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}1&L&D+\frac{1}{2}LR\\0&1&R\\0&0&1\end{bmatrix}. \tag{66}$$

Computing the expressions on the left and the right of equation (65) in this representation, we find

$$\begin{bmatrix}1&\beta&\frac{1}{2}\alpha\beta\\0&1&\alpha\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&\alpha'\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&d'\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&\beta'&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&\beta'&d'\\0&1&\alpha'\\0&0&1\end{bmatrix}. \tag{67}$$

By comparing the expressions on the left and the right, matrix element by matrix element, we conclude: $\alpha'=\alpha, \beta'=\beta, d'=\frac{1}{2}\alpha\beta$. Formally, we have done the following [27, 29]

$$e^{\Gamma^f(\alpha a^\dagger+\beta a)}=e^{\alpha\Gamma^f(a^\dagger)}\ e^{\frac{1}{2}\alpha\beta\Gamma^f(I)}\ e^{\beta\Gamma^f(a)} \Leftrightarrow e^{\alpha a^\dagger+\beta a}=e^{\alpha a^\dagger}\ e^{\frac{1}{2}\alpha\beta\ I}\ e^{\beta a}. \tag{68}$$

To illustrate how *disentangling formulas* of this type are useful, we observe that $x, D=\mathrm{d}/\mathrm{d}x, I$ span $\mathfrak{h}_3$, so that we can use the method above to construct the formula

$$e^{t(x-D)}=e^{tx}e^{-t^2/2}e^{-tD}. \tag{69}$$

This can be applied to the Gaussian function $e^{-x^2/2}$ in two different ways

$$e^{t(x-D)}e^{-x^2/2}=e^{-x^2/2}\sum_{n=0}\frac{t^nH_n(x)}{n!}. \tag{70}$$

This is a Rodrigues' expression for Hermite polynomials [1]. An alternative application is

$$e^{tx}e^{-t^2/2}e^{-tD}e^{-x^2/2}=e^{tx}e^{-t^2/2}e^{-(x-t)^2/2}=e^{2xt-t^2}. \tag{71}$$

The disentangling theorem for $\mathfrak{h}_3$ leads in a simple way to the generating function for Hermite polynomials [1].

12.2. Heisenberg Solvable Group

The Heisenberg group H_4 is a four-dimensional solvable Lie group [29]. The four basis operators in $\mathfrak{h}_4$ include besides those of $\mathfrak{h}_3$ the so called number operator $\hat{n} = \frac{1}{2}(aa^\dagger + a^\dagger a)$, with two additional nonzero commutators: $\left[\hat{n}, a^\dagger\right] = +a^\dagger$ and $[\hat{n}, a] = -a$. This algebra also acts in the infinite-dimensional Hilbert space described above. It also has a faithful 3×3 matrix representation [29]

$$\Gamma^f(N\hat{n} + La + Ra^\dagger + DI) = \begin{bmatrix} 0 & L & D \\ 0 & N & R \\ 0 & 0 & 0 \end{bmatrix}. \tag{72}$$

The exponential of this matrix representation is a bit more complicated

$$\begin{aligned} \mathrm{Exp}\left(\Gamma^f(N\hat{n} + La + Ra^\dagger + DI)\right) &\to \mathrm{Exp}\left(\begin{bmatrix} 0 & L & D \\ 0 & N & R \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & \frac{L}{N}(\mathrm{e}^N - 1) & D + LR\frac{\mathrm{e}^N - 1 - N}{N^2} \\ 0 & \mathrm{e}^N & \frac{R}{N}(\mathrm{e}^N - 1) \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{73}$$

This expression can be used to construct all sorts of disentangling results with relative ease.

It happens often that thermal expectation values $\langle \mathcal{O} \rangle$ need to be constructed. These can be constructed from generating function $\langle \mathrm{e}^{\lambda\mathcal{O}} \rangle = F(\mathcal{H}, \lambda)/F(\mathcal{H}, 0)$, where $F(\mathcal{H}, \lambda) = \mathrm{tr}\, \mathrm{e}^{-\beta\mathcal{H}}\mathrm{e}^{\lambda\mathcal{O}}$, by taking derivatives

$$\langle \mathcal{O}^n \rangle = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \langle \mathrm{e}^{\lambda\mathcal{O}} \rangle|_{\lambda=0}\ . \tag{74}$$

It happens often under suitable approximations that the operators $\mathcal{O}$ are elements in a finite-dimensional Lie algebra and the Hamiltonian $\mathcal{H}$ is also a linear element in the Lie algebra. In such cases the product $\mathrm{e}^{-\beta\mathcal{H}}\mathrm{e}^{\lambda\mathcal{O}}$ is an element in a Lie group and the theorem described above becomes very useful.

In order to illustrate how this works, suppose we needed to compute thermal expectation values of x^2, p^2, or $xp + px$ in a harmonic oscillator potential under thermal equilibrium conditions. Then $\mathcal{H} = \hbar\omega\hat{n}$ and we can choose $\mathcal{O} = Ra^\dagger + La$ and attempt to compute $\mathrm{e}^{-\beta\hbar\omega\hat{n}}\mathrm{e}^{\lambda(Ra^\dagger + La)}$. In the infinite-dimensional representation this would be a nightmare but in the 3×3 faithful matrix representation it is not overly difficult

$$\mathrm{e}^{-\beta\hbar\omega\hat{n}}\mathrm{e}^{\lambda(Ra^\dagger + La)} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-\beta\hbar\omega} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda L & \frac{1}{2}\lambda^2 LR \\ 0 & 1 & \lambda R \\ 0 & 0 & 1 \end{bmatrix}. \tag{75}$$

Furthermore, there are a few extra tricks that greatly simplify the computation. The first is that the trace of a matrix is invariant under a similarity transformation. The second is that $\hat{n}$ and I are diagonal in the infinite-dimensional representation of use in Quantum Theory, even if I is *not* diagonal in the faithful 3×3 finite-dimensional representation. (Recall: Being diagonal or unitary is a property of the representation of the group/algebra, not of the group/algebra itself.) Therefore it would be useful to hunt for a similarity transformation that transforms the product in equation (75) to the exponential of just the two operators $\hat{n}$ and I. Introduce the group operation $S = \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and its inverse and apply it to find

$$\begin{aligned} & S \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-\beta\hbar\omega} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda L & \frac{1}{2}\lambda^2 LR \\ 0 & 1 & \lambda R \\ 0 & 0 & 1 \end{bmatrix} S^{-1} \\ &= \begin{bmatrix} 1 & b(-1+\mathrm{e}^{-\beta\hbar\omega}) + \lambda L & ** \\ 0 & \mathrm{e}^{-\beta\hbar\omega} & c(1-\mathrm{e}^{-\beta\hbar\omega}) + \mathrm{e}^{-\beta\hbar\omega}\lambda R \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{76}$$

The values of b and c are now chosen to zero out the coefficients of the creation and annihilation operators, and $**$ evaluates to

$$** = \frac{1}{2}\lambda^2 LR \coth(\frac{\hbar\omega}{2kT}). \tag{77}$$

This disentangling result is now used in the infinite-dimensional representation, giving

$$\begin{aligned} F(\mathcal{H}, \lambda) = \operatorname{tr} \mathrm{e}^{-\beta\hbar\omega\hat{n} + **} &= \mathrm{e}^{\frac{1}{2}\lambda^2 LR \coth(\frac{\hbar\omega}{2kT})} \operatorname{tr} \mathrm{e}^{-\beta\hbar\omega\hat{n}} \\ &= \mathrm{e}^{\frac{1}{2}\lambda^2 LR \coth(\frac{\hbar\omega}{2kT})} \langle \hat{n} \rangle_T. \end{aligned} \tag{78}$$

As usual $\langle n \rangle_T = \frac{1}{2} + \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}$ and the final generating function is

$$\langle \mathrm{e}^{\lambda(Ra^\dagger + La)} \rangle = \mathrm{e}^{\frac{1}{2}\lambda^2 LR \coth(\frac{\hbar\omega}{2kT})}. \tag{79}$$

Various choices of the coefficients L, R allow computation of the moments of various powers of $x, p, xp + px$. For example, with $L = R = 1/\sqrt{2}$, $(Ra^\dagger + La) = x$ so that $\langle \mathrm{e}^{\lambda x} \rangle = \mathrm{e}^{\frac{1}{4}\lambda^2 \coth(\frac{\hbar\omega}{2kT})}$. It is clear that the odd moments of x vanish (by symmetry) and the first even moment is $\langle x^2 \rangle = \frac{1}{2} \coth(\frac{1}{2}\beta\hbar\omega)$ [26].

12.3. SU(2) Coherent State Parameterization

Coherent states for the harmonic oscillator were first introduced by Schrödinger in 1926 in his attempt to understand the relation between his new creation, Quantum Mechanics (Wave Mechanics) and Classical Mechanics [51]. They were later rediscovered by many people. In the early 1960s they were used by Glauber [30, 31] to create a profoundly powerful foundation for the field of Quantum Optics, spurred by the recent development of masers and lasers.

Masers and lasers involve atoms and fields interacting in useful ways. The Glauber-Schrödinger coherent states were applied to describe the field part of these devices. In the late 1960s the question was raised whether something like coherent states could also be formulated to describe the atomic side of these devices. Specifically, could coherent states be constructed for atoms involved in lasing transitions? In effect the question was directed at describing the atomic part of the transition induced by a single frequency mode of the electromagnetic field, which coupled two atomic levels. Such atomic transitions could be described by two shift operators, describing transitions from the lower *ground* to the higher *excited* state (σ_+) and in the reverse direction (σ_-), as well as two operators describing the two energy eigenstates, $\frac{1}{2}(I_2 \pm \sigma_3)$. The dynamics of a two-level atom could be described by the operators of the unitary algebra $\mathfrak{u}(2)$.

When N atoms are present each atom is described by operators $\sigma^{(i)}_{\pm,3,0}$. When all atoms act coherently the individual atomic operators can be summed over to give angular momentum operators and the identity: $J_{\pm,3,0} = \sum_{i=1}^{N} \sigma^{(i)}_{\pm,3,0}$. These collective operators obey the usual commutation relations $[J_3, J_\pm] = \pm J_\pm$, $[J_+, J_-]$ $= 2J_3$, with I_{N+1} commuting with the three collective angular momentum operators.

Since the field coherent states for a single mode involved four operators $\hat{n}, a^\dagger, a, I$ and the collective atomic states were also described by four operators with analogous but somewhat different properties, it was felt that it could be possible to create atomic coherent states that shared a similar spectrum of properties as the field coherent states [3]. In fact, the four field operators can be obtained by a certain contraction limit from the four atomic operators [3, 26].

Field coherent states for a single mode could be defined from three different starting points [30, 31]

1. As minimum uncertainty states.

2. As eigenstates of the annihilation operator.

3. By applying a group transformation from H_4 to the ground state.

These three approaches were equivalent because of the structure of the Lie algebra $\mathfrak{h}_4$. Neither of the first two approaches could be used to create coherent states for a finite number of two-level atoms. This is directly due to the fact that the atomic Hilbert space is finite dimensional. In the case that N atoms are present, all initially in their ground state, the Hilbert space is defined by $J = \frac{1}{2}N$ and the ground state is $|\mathrm{ground}\rangle = |\begin{smallmatrix} J \\ -J \end{smallmatrix}\rangle$

Atomic coherent states were developed using the construction of field coherent states as a model [3]. On the field side the ground state $|0\rangle$ is left invariant (effectively unchanged, multiplied by a phase factor) by exponentials of the number and identity operators. On the atomic side the ground state is multiplied by a phase factor under action by exponentials of the operators J_3, I_{2J+1}. In summary

$$\mathrm{e}^{\gamma\hat{n}+\delta I}|0\rangle = \mathrm{phase} \times |0\rangle, \qquad \mathrm{e}^{\gamma J_3+\delta I_{2J+1}}|\begin{smallmatrix} J \\ -J \end{smallmatrix}\rangle = \mathrm{phase} \times |\begin{smallmatrix} J \\ -J \end{smallmatrix}\rangle. \tag{80}$$

The field coherent states are created by a rotation using the two remaining operators and the atomic coherent states could be similarly created as follows

$$|\alpha\rangle = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}|0\rangle, \qquad |\theta,\phi\rangle = \mathrm{e}^{-i\theta(J_x \sin\phi - J_y \cos\phi)}|\begin{smallmatrix} J \\ -J \end{smallmatrix}\rangle. \tag{81}$$

The field coherent states could be expressed in terms of the number (Fock) states $|n\rangle$ using a distentangling theorem; so also could the atomic coherent states be expressed in terms of eigenstates $|\begin{smallmatrix} J \\ M \end{smallmatrix}\rangle$ of J_3. The comparison is more direct by making the change of variables $\mathrm{e}^{-\mathrm{i}\theta(J_x \sin\phi - J_y \cos\phi)} = \mathrm{e}^{\zeta J_+ - \zeta^* J_-}$ with $\zeta = (\theta/2)\mathrm{e}^{-\mathrm{i}\phi}$

$$\begin{aligned} U(\alpha) &= \mathrm{e}^{\alpha a^\dagger - \alpha^* a} = \mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^*\alpha/2I}\mathrm{e}^{-\alpha^* a} \\ U(\theta,\phi) &= \mathrm{e}^{\zeta J_+ - \zeta^* J_-} = \mathrm{e}^{\tau J_+}\mathrm{e}^{\ln(1+\tau^*\tau)J_3}\mathrm{e}^{-\tau^* J_-} \end{aligned} \tag{82}$$

where τ and ζ are projectively related and $\tau = \mathrm{e}^{-\mathrm{i}\phi}\tan\frac{1}{2}\theta$. By applying the disentangled operators in equation (83) to the ground states, the coherent states are obtained as linear combinations of the eigenstates of $\hat{n}$ and J_3

$$\begin{aligned} |\alpha\rangle &= U(\alpha)|0\rangle = \mathrm{e}^{-\alpha^*\alpha/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\ |\theta,\phi\rangle &= U(\theta,\phi)|\begin{smallmatrix} J \\ -J \end{smallmatrix}\rangle = \sum_{M=-J}^{M=+J}\left(\frac{2J}{J+M}\right)^{1/2}\frac{\tau^{J+M}}{(1+\tau^*\tau)^J}|\begin{smallmatrix} J \\ M \end{smallmatrix}\rangle. \end{aligned} \tag{83}$$

Using these expansions it is possible to compute many useful propeerties, for example overlaps

$$\langle\alpha|\beta\rangle = \mathrm{e}^{-(|\alpha|^2+|\beta|^2-2\alpha^*\beta)/2}, \qquad \langle\theta,\phi|\theta',\phi'\rangle = \left(\frac{(1+\tau^*\tau')^2}{(1+|\tau|^2)(1+|\tau'|^2)}\right)^J \tag{84}$$

and their absolute squares

$$|\langle\alpha|\beta\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}, \qquad |\langle\theta,\phi|\theta',\phi'\rangle|^2 = \cos^{4J}\frac{1}{2}\Theta \tag{85}$$

where the angle Θ is defined in the usual way as the distance between two points on a unit sphere with coordinates (θ,ϕ) and (θ',ϕ'), i.e., $\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$.

Since the development of atomic coherent states depends on the properties of groups, their representations, and analytic reparameterizations, it has become possible to construct coherent states for any group. The prescription is standard and follows the algorithm presented above.

12.4. Rotation Group Parameterizations

Many different parameterizations of the rotation group $\mathrm{SO}(3)$ have been introduced in the past [6, 46, 48]. These have often been tailored to specific needs in widely different fields. As a result there is a wide – possibly confusing – array of representations of the group composition law. Carried out correctly, they are all equivalent. Often it is difficult to see their equivalences or the relations among them.

The rotation group has been parameterized by Euler angles, Rodrigues vectors, Cayley-Klein parameters, Fedorov-Gibbs vectors, Cayley transformations as well as the exponential map [2, 7, 11, 13, 16, 33, 34, 38, 39, 42–45, 47, 52, 53, 58]. The disentangling theorem is a useful means to create analytic mappings among the large variety of different analytic parameteriztions of the rotation group.

To begin, it is useful to adopt the smallest faithful matrix representation of this group. This is the 2×2 spinor representation, which is not only faithful, but a double cover of the rotation group. We use the exponential of elements in the Lie algebra as a Rosetta stone for comparing the myriad different analytic descriptions of this group. A rotation about an axis $\hat{\mathbf{n}}$ in $\mathbb{R}^3$ through an angle θ is represented by the 2×2 matrix

$$\mathrm{e}^{\mathrm{i}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\theta/2} = \begin{bmatrix} \cos(\theta/2)+\mathrm{i}n_3\sin(\theta/2) & (\mathrm{i}n_1+n_2)\sin(\theta/2) \\ (\mathrm{i}n_1-n_2)\sin(\theta/2) & \cos(\theta/2)-\mathrm{i}n_3\sin(\theta/2) \end{bmatrix}. \tag{86}$$

Euler-angle type parameterizations are characterized by a succession of three successive nonparallel rotation axes. For example, an Euler-angle parameterization of type 3-1-2 involves a succession of rotations, first around the second axis (Y), followed by a rotation around the first axis, concluding with a rotation about the Z-axis. Suppose we want to describe a rotation in terms of a 3-1-3 Euler parameteriztion [32] $R_Z(\phi_2)R_X(\psi)R_Z(\phi_1)$. The spinor representation of this sequence of rotations is

$$\begin{aligned}
&\begin{bmatrix} \mathrm{e}^{\mathrm{i}\phi_2/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi_2/2} \end{bmatrix} \begin{bmatrix} \cos(\psi/2) & \mathrm{i}\sin(\psi/2) \\ \mathrm{i}\sin(\psi/2) & \cos(\psi/2) \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\phi_1/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi_1/2} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{\mathrm{i}(\phi_2+\phi_1)/2}\cos(\psi/2) & \mathrm{e}^{\mathrm{i}(\phi_2-\phi_1)/2}\mathrm{i}\sin(\psi/2) \\ \mathrm{e}^{\mathrm{i}(-\phi_2+\phi_1)/2}\mathrm{i}\sin(\psi/2) & \mathrm{e}^{\mathrm{i}(-\phi_2-\phi_1)/2}\cos(\psi/2) \end{bmatrix}.
\end{aligned} \tag{87}$$

The analytic transformation between the exponential parameterization in equation (86) and the Euler 3-1-3 parameterization of equation (88) is obtained simply by comparing these 2×2 matrices, matrix element by matrix element. This is a two-way street. Either triplet of parameters $(\hat{\mathbf{n}}, \theta)$ or (ϕ_2, ψ, ϕ_1) can be expressed analytically in terms of the other. For example, if the exponential parameterization is known and it is desired to determine the sequence of Euler rotations leading to this particular group operation, it is sufficient to solve the equations

$$\begin{aligned}
\tan(\tfrac{1}{2}(\phi_2 + \phi_1)) &= n_3 \tan\tfrac{1}{2}\theta \\
\tan(\tfrac{1}{2}(\phi_2 - \phi_1)) &= -n_2/n_1 \\
\cos(\psi/2) &= \sqrt{\cos^2(\theta/2) + (n_3\sin(\theta/2))^2}.
\end{aligned} \tag{88}$$

12.5. SU(4) Coherent States

The dynamics of an ensemble of identical four-level atoms driven by a classical field can be described by $\mathrm{SU}(4)$ coherent states. These are constructed following the algorithm described above for atomic (or $\mathrm{SU}(2)$) coherent states [4, 24, 25, 28]. For simplicity we construct $\mathrm{U}(4)$ coherent states, which differ from $\mathrm{SU}(4)$ coherent states by an unimportant overall phase factor.

First, assume that each atom has four states of interest, $|k\rangle$, $k = 1, 2, 3, 4$, with $|1\rangle$ being the ground state. Also assume that at some time all N atoms are in their respective ground states. Then in the future the collective state will be a linear combination of states of the form $|n_1, n_2, n_3, n_4\rangle$, with $n_1 + \cdots + n_4 = N$. The total number of states of this type is $\begin{pmatrix} N+3 \\ 3 \end{pmatrix} = (N+3)(N+2)(N+1)/6$. This is the dimensionality of the symmetric N^{th} order tensor product of the defining matrix representation $\mathbf{F}_1$ of $\mathrm{SU}(4)$.

It is useful to define the 16 infinitesimal generators in the Lie algebra $\mathfrak{u}(4)$ as $X_{ji} = b_j^\dagger b_i$, where X_{ji} shifts an excitation from state i to state j. The operators X_{ii} are "energy operators": they determine the number of atoms in the i^{th} state.

The U(4) coherent states are obtained by applying a general U(4) transformation on the ground state $|\text{ground}\rangle = |N, 0, 0, 0\rangle$. It is clear that the U(3) subgroup of operators of the form $\mathrm{Exp}(\sum_{rs} \alpha_{rs} b_r^\dagger b_s)$, with $2 \leq r, s \leq 4$, leave the ground state invariant. The rotations that produce $\mathrm{U}(4)$ coherent states are drawn from the coset U(4)/U(3)× U(1). These group operations have the form $\mathrm{Exp}\Big(\sum_r (\zeta_r b_r^\dagger b_1 - \zeta_r^* b_1^\dagger b_r)\Big)$. A disentangling theorem of the form

$$\mathrm{e}^{\zeta_r b_r^\dagger b_1 - \zeta_r^* b_1^\dagger b_r} = \mathrm{e}^{\tau_r b_r^\dagger b_1} \mathrm{e}^{\alpha_{rs} b_r^\dagger b_s + \alpha_{11} b_1^\dagger b_1} \mathrm{e}^{-\tau_r^* b_1^\dagger b_r} \tag{89}$$

could then be used to construct the U(4) coherent states

$$\mathrm{e}^{\zeta_r b_r^\dagger b_1 - \zeta_r^* b_1^\dagger b_r} |N, 0, 0, 0\rangle = \mathrm{e}^{\tau_r b_r^\dagger b_1} \mathrm{e}^{N\alpha_{11}} |N, 0, 0, 0\rangle. \tag{90}$$

The three operators in the exponential on the right hand side commute, so they can be applied independently. Carrying out this expansion, we find

$$|\zeta\rangle = \sum_{n_2} \sum_{n_3} \sum_{n_4} \left(\frac{N!}{n_1! n_2! n_3! n_4!} \right)^{1/2} \mathrm{e}^{N\alpha_{11}} \tau_2^{n_2} \tau_3^{n_3} \tau_4^{n_4} |n_1, n_2, n_3, n_4\rangle \tag{91}$$

with $n_1 + n_2 + n_3 + n_4 = N$ and $n_i \geq 0$.

It only remains to determine the analytical relation between the initial coherent state parameters ζ and the disentangling parameters τ, α_{11}. This is carried out by matrix multiplication in the defining 4×4 matrix representation of $\mathfrak{u}(4)$. Explicitly, we compute

$$\mathrm{Exp}\left[\begin{array}{c|c} 0 & \boldsymbol{\zeta} \\ \hline -\boldsymbol{\zeta}^\dagger & 0 \end{array}\right] = \mathrm{Exp}\left[\begin{array}{c|c} 0 & \boldsymbol{\tau} \\ \hline 0 & 0 \end{array}\right] \mathrm{Exp}\left[\begin{array}{c|c} M(3) & 0 \\ \hline 0 & M(1) \end{array}\right] \mathrm{Exp}\left[\begin{array}{c|c} 0 & 0 \\ \hline -\boldsymbol{\tau}^\dagger & 0 \end{array}\right]. \tag{92}$$

In these expressions $\boldsymbol{\tau}$ is a 3×1 column vector $\begin{pmatrix} \tau_4 & \tau_3 & \tau_2 \end{pmatrix}^t$ and $M(1)$ is a 1×1 matrix. Carrying out the indicated computations provides the desired relations

$$|\zeta| = \sqrt{\boldsymbol{\zeta}^\dagger \boldsymbol{\zeta}}\,, \qquad M(1) = \mathrm{e}^{\alpha_{11}} = \cos|\zeta|\,, \qquad \boldsymbol{\tau} = \frac{\tan|\zeta|}{|\zeta|}\,\boldsymbol{\zeta}. \tag{93}$$

The inner product is

$$\langle\zeta|\zeta'\rangle \to \langle\tau|\tau'\rangle = \left(\frac{(1+\boldsymbol{\tau}^{\dagger}\boldsymbol{\tau}')}{(1+\boldsymbol{\tau}^{\dagger}\boldsymbol{\tau})^{1/2}(1+\boldsymbol{\tau'}^{\dagger}\boldsymbol{\tau}')^{1/2}}\right)^{N}. \tag{94}$$

The construction of U(4) coherent states for other than the symmetric class of representations is also possible [24, 27]. However, it involves more work and is less important, as the enlarged class of representations do not contain the natural physical ground state.

13. Invariant Operators

Invariant operators are functions of the generators of a Lie group that commute with all operators in the Lie algebra. A typical example of an invariant operator is the square of the total angular momentum $\mathbf{J}\cdot\mathbf{J}$ for SU(2). If $F(X)$ is an invariant operator, $[X, F(X)] = 0$ for all operators X in the Lie algebra.

Alternatively, an invariant operator commutes with all operations in the Lie group: $gF(X)g^{-1} = F(X)$. The equivalence is easily seen by writing $g = g(\lambda) = \mathrm{e}^{\lambda X}$ and then taking the limit of $\mathrm{e}^{\lambda X}F(X)\mathrm{e}^{-\lambda X} \to F(X) + \lambda X F(X) - F(X)(\lambda X)$ $+$higher order terms $= F(X)$.

13.1. Casimir Invariants

A systematic way for constructing quadratic invariant operators for semisimple Lie algebras was devised by Casimir [10]. For a semisimple Lie algebra the structure constants ${C_{ij}}^{k}$ are defined in terms of the commutation relations by $[X_i, X_j] = {C_{ij}}^{k}X_k$. The structure constants are components of a third order tensor, first order contravariant and second order covariant. It is antisymmetric in the covariant indices. A covariant second order symmetric tensor can be constructed by double cross contraction

$$G_{ij} = \sum_{r,s} {C_{ir}}^{s}{C_{js}}^{r} = G_{ji}. \tag{95}$$

This tensor is nonsingular if and only if the underlying Lie algebra is semisimple [26]. Its inverse, G^{ij}, gives a second order invariant operator when contracted against the operators X_i

$$F(X) = G^{ij}X_iX_j, \qquad [X_r, F(X)] = 0. \tag{96}$$

The proof is straightforward and depends on the antisymmetry of the structure constants with two contravariant indices: ${C_i}^{rs} + {C_i}^{sr} = 0$.

It is useful to investigate the invariance of this Casimir operator under group transformations

$$g(G^{ij}X_iX_j)g^{-1} = G^{ij}(gX_ig^{-1})(gX_jg^{-1}) = G^{ij}\Gamma^{\mathrm{reg}}_{ir}(g)X_r\Gamma^{\mathrm{reg}}_{js}(g)X_s \;. \quad (97)$$

Here $\Gamma^{\mathrm{reg}}(g)$ is the *regular* matrix representation of the Lie group/algebra. It consists of $n \times n$ matrices, where n is the dimension of the group/algebra. The regular representation is usually neither irreducible nor fundamental but is easy to construct in terms of the structure constants for the Lie algebra: $\Gamma^{\mathrm{reg}}_{jk}(X_i) = C_{ij}{}^{k}$. Since $G^{ij}X_iX_j$ is invariant under g, equation (97) requires

$$G^{ij}\Gamma^{\mathrm{reg}}_{ir}(g)\Gamma^{\mathrm{reg}}_{js}(g) = G^{rs}, \qquad \Gamma^{\mathrm{reg}\;t}(g)G\Gamma^{\mathrm{reg}}(g) = G \;. \quad (98)$$

The regular representation of a semisimple Lie group is a metric-preserving representation. It preserves the Cartan-Killing metric.

13.2. Casimir Covariants

If Γ^a and Γ^b are two representations of a semisimple Lie algebra and its Lie group, the operator constructed on the tensor product of the spaces carrying these two representations is also an invariant operator

$$\begin{aligned} gF(X;a,b)g^{-1} &= G^{ij}\Gamma^a(gX_ig^{-1})\Gamma^b(gX_jg^{-1}) \\ &= G^{ij}\Gamma^{\mathrm{reg}}_{ri}(g)\Gamma^a(X_r)\Gamma^{\mathrm{reg}}_{sj}(g)\Gamma^b(X_s) = G^{rs}\Gamma^a(X_r)\Gamma^b(X_s). \end{aligned} \quad (99)$$

The last equation used the metric-preserving condition established in equation (98). Such operators are called Casimir covariants.

13.3. Other Invariant Operators

Simple Lie groups of rank r have r functionally independent invariant operators. The product of the orders of these invariant operators is equal to the order of the Weyl group of the Lie algebra [56]. This is the discrete group of operations that maps the root vectors onto themselves.

The invariant operators can be determined in a number of simple ways [22, 26]. One is to compute the *secular equation* for a general element of the Lie algebra in the regular representation. For a general element in $\mathfrak{so}(3)$ (cf equation (21))

$$\boldsymbol{\theta}\cdot\mathbf{L} \rightarrow \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \xrightarrow{\text{sec. equation}} (-\lambda)^3 + (-\lambda)(\theta_1^2+\theta_2^2+\theta_3^2) \;. \quad (100)$$

The substitution $\theta_i \rightarrow L_i$ transforms each functional coefficient in the secular equation into an invariant operator.

This calculation can be simplified by taking a page from the reparameterization algorithm discussed in Section 12. That is, the secular equation for the smallest faithful matrix representation of the Lie aglebra provides this information. As an example, the spinor representation of the Lorentz group, given in equation (40), yields a quadratic coefficient with real part leading to one quadratic invariant $F_1(\mathbf{L},\mathbf{B}) = \mathbf{L}\cdot\mathbf{L} - \mathbf{B}\cdot\mathbf{B}$ and imaginary part leading to another independent quadratic invariant $F_2(\mathbf{L},\mathbf{B}) = \mathbf{L}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{L}$.

The general invariant operator can be expressed in the form

$$F^k(X) = G^{i,j,\ldots,k} X_i X_j \ldots X_k. \tag{101}$$

The k^{th} order tensor $G^{i,j,\ldots,k}$ is symmetric. Covariant operators can be constructed from this invariant in an obvious way

$$F^k(X; a, b, \ldots, c) = G^{i,j,\ldots,k} \Gamma^a(X_i)\Gamma^b(X_j)\ldots\Gamma^c(X_k). \tag{102}$$

The most familiar of the covariant operators is the spin-orbit coupling operator $\mathbf{L}\cdot\mathbf{S}$, widely used in the study of atomic spectroscopy.

14. Discussion

Before Quantum Mechanics was developed, Physicists were conversant with the rotation group $\mathrm{SO}(3)$ and even the tensor representations based on the defining three-dimensional representation acting on $\mathbf{r}$, the coordinates of a vector in $\mathbb{R}^3$. The rank-l irreducible tensors had dimension $2l + 1$ and were widely known. The discovery of spin came like a thunderbolt, as the *spinor* representations of $\mathrm{SO}(3)$ had not been known (to Physicists). At the level of the Lie algebra, $\mathfrak{so}(3) = \mathfrak{su}(2)$ and the spin representations of $\mathrm{SO}(3)$ are the vector representations of $\mathrm{SU}(2)$. The representations ($\mathcal{D}^j(\mathrm{SO}(3))$ in Wigner's notation [57]) were tensor representations of $\mathrm{SU}(2)$ of rank $2j$, $2j = 0, 1, 2, \cdots$ and dimension $2j + 1$. For $j = l$ (integer) these Wigner rotation matrices could be made real but for j half-integer they are essentially complex and cannot be constructed as tensor products of representations based on three-vectors $\mathbf{r}$ alone.

The relation among the simple Lie algebras of low rank and their fundamental irreducible representations has reduced the level of mystery surrounding the so-called spinor representations of higher rank groups. For $D_2 = A_1 \oplus A_1$ there are two fundamental irreducible representations, each of dimension two, one for each copy of A_1. Any real form of $\mathfrak{d}_2$ has two distinct, inequivalent spinor representations. One of the real forms is $\mathfrak{so}(3,1)$, the Lorentz group. One should therefore expect that these two spinor representations might enter relativistic physics in interesting and exciting ways. It has been customary to represent the tensor product as $\mathbf{F}_1^{j_1} \otimes \mathbf{F}_2^{j_2} \simeq \mathcal{D}^{(j_1,j_2)}$. Then $\mathcal{D}^{(\frac{1}{2},\frac{1}{2})} \simeq \mathrm{SO}(3,1)$ and the Dirac wavefunction, a column spinor with 4 = 2+2 components, carries the representation $\mathcal{D}^{(\frac{1}{2},0)} + \mathcal{D}^{(0,\frac{1}{2})} = \mathcal{D}^{(\frac{1}{2},0)+(0,\frac{1}{2})}$ [36]. The components of the spinors belonging to $\mathbf{F}_1$ were distinguished from those belonging to $\mathbf{F}_2$ by placing a dot over the indices of the latter (hence, *dotted* spinors [12]). The components $\mathbf{E}, \mathbf{B}$ of the electromagnetic field carry the representation $\mathcal{D}^{(1,0)+(0,1)}$.

For $\mathrm{SO}(5)$ the first fundamental representation is five-dimensional. The pth order tensor products based on this representation offer no surprises. Their highest weights are $p\mathbf{e}_1$ (p integer) and all can be made real. However, its second fundamental representation is four dimensional and cannot be made real. The qth order tensor products based on this representation have highest weights $\frac{q}{2}(\mathbf{e}_1 + \mathbf{e}_2)$. When q is odd all weights in this representation have half-integer values. In this sense $\mathbf{F}_2$ is the spinor representation of $\mathrm{SO}(5)$ and all its real forms. For $\mathrm{Sp}(4,\mathbb{R}) \simeq C_2$ the first fundamental representation $\mathbf{F}_1$ is four-dimensional and therefore the spin representation of B_2. The second fundamental irreducible representation $\mathbf{F}_2$ of C_2 is five-dimensional and therefore equivalent to the vector representation of B_2.

For $\mathrm{SU}(4) \simeq A_3$ the first fundamental representation is four dimensional. A basis vector for this representation is a complex four-component vector in the four-dimensional Hilbert space on which $\mathrm{SU}(4)$ acts. The third fundamental representation is also four-dimensional. This representation is the complex conjugate of the first fundamental representation. The second fundamental representation is six-dimensional. It acts on an antisymmetric second order tensor based on four-vectors. This representation can be made real.

On the orthogonal side of the equivalence $A_3 = D_3$ the first fundamental representation of $\mathrm{SO}(6)$ is six-dimensional and can be made real. The two other fundamental representations are both four-dimensional. They are two inequivalent spinor representations of $\mathrm{SO}(6)$. Each of these representations is equivalent to one of the four-dimensional representations of $\mathrm{SU}(4)$. The two spinor representations of $\mathrm{SO}(6)$ are complex conjugates of each other. The Lie group $\mathrm{SO}(6)$ and its noncompact real forms appear in Physics in many different ways [26, 29]. For this reason it is useful to understand the relation between the real forms of $\mathrm{SO}(6)$ and those of $\mathrm{SU}(4)$, as well as the relations among their fundamental irreducible representations.

Lie groups can be parameterized in many different ways. This is particularly noticeable for the rotation group, which has seen physical and engineering applications for over two hundred years. Each different parameterization induces a different analytic representation for the group multiplication and inversion rules. Relating different parameterizations, and their composition and inversion laws, can be somewhat of a nightmare. A simple algorithm for carrying out all such computations was introduced and applied to a number of useful cases. We have closed by introducing invariant and covariant operators, and shown how to compute them using the *reparameterization trick*.

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